Chapter 1: Poisson processes

Ref: Chapter 5 of V.G. Kulkarni's book

Exponential distribution

A r.v. $X \sim Exp(\lambda)$, $\lambda > 0$ if its CDF

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

Density of X is given by

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

Laplace Stieltjes transform (LST)

$$E(e^{-sX}) \leftarrow LST$$

For $X \sim Exp(\lambda)$,

$$\phi_X(s) = E(e^{-sX}) = \int_0^\infty e^{-sx} \lambda e^{-\lambda x} dx$$

① → $\phi_X(s) = \dfrac{\lambda}{\lambda+s}$, $\mathrm{Re}(s) > -\lambda$

From ①, $E(X^r) = (-1)^r \dfrac{d^r}{ds^r} \phi_X(s) \Big|_{s=0}$

$$E(X^r) = \frac{r!}{\lambda^r}$$

Special cases: $E(X) = \dfrac{1}{\lambda}$, $Var(X) = \dfrac{1}{\lambda^2}$

"LST uniquely determines the distribution"

Memoryless property:

A r.v. X is "memoryless" if

$$P(X > s+t \mid X > s) = P(X > t), \quad \forall s,t \geq 0$$

Example: If X models the lifetime of an item, say tubelight.

Then, by memorylessness we are saying
"Prob(A s-day old tubelight lasting for another t days)
= Prob(A new tubelight lasting for t days)"

Theorem:

A continuous r.v. $X \geq 0$ is memoryless if and only if it is $\mathcal{E}xp(\lambda)$ for some $\lambda > 0$

Proof: ($\Rightarrow$) Suppose $X \sim \mathcal{E}xp(\lambda)$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t,\ X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$P(X > s+t \mid X > s) = P(X > t)$$

So, X is "memoryless".

($\Leftarrow$) Suppose X is memoryless

Let $F_c(x) = P(X > x)$ be the CCDF

Given $F_c(s+t) = F_c(s)\, F_c(t)$, $\forall s, t \geq 0$

$$F_c(2) = F_c(1)^2$$

$$F_c\left(\tfrac{1}{2}\right) = \sqrt{F_c(1)}$$

For any rational number, say $\frac{p}{q}$,

$$F_c\left(\frac{p}{q}\right) = \left(F_c(1)\right)^{p/q}$$

Taking limit of rational numbers, we have

$$F_c(x) = \left(F_c(1)\right)^x \quad , \quad \forall x \in \mathbb{R}^+$$

So, what we have is

$$F_c(x) = \left(F_c(1)\right)^x$$
$$= \exp\left(x \ln F_c(1)\right)$$

$$0 < F_c(1) \leq 1$$
$$\Rightarrow \quad \lambda = -\ln F_c(1) \quad , \lambda > 0$$

$$F_c(x) = \exp(-\lambda x) \, , \quad \lambda > 0$$

Hence, $\quad X \sim \mathcal{E}xp(\lambda)$

■

EXAMPLES:

① $\quad X \sim \mathcal{E}xp(\lambda) \, , \quad Y \sim \mathcal{E}xp(\mu)$

X, Y independent $\quad f_{X,Y}(x,y) = \underbrace{\lambda e^{-\lambda x} \mu e^{-\mu y}}_{\text{since } X, Y \text{ indep.}}$

(i) $\quad$ Let $\quad Z = \min(X, Y)$

$$P(Z > z) = P(X > z, Y > z)$$

$$= P(X > z)\, P(Y > z)$$
$$= e^{-\lambda z}\, e^{-\mu z}$$
$$= e^{-(\lambda+\mu) z}, \quad \forall z \geq 0$$

So, $\quad Z \sim \mathcal{E}xp(\lambda + \mu)$

(ii) $\quad P(X < Y) = \int_0^\infty \left(\int_0^y \lambda e^{-\lambda x}\, dx \right) \mu e^{-\mu y}\, dy$

$$= \int_0^\infty \left(1 - e^{-\lambda y}\right) \mu e^{-\mu y}\, dy$$

$$= \frac{\lambda}{\lambda + \mu}.$$

(iii) Minimum of "n" exponentials:

$\{X_1, \ldots X_n\}$ independent

$X_i \sim \mathcal{E}xp(\lambda_i)$

Let $\quad Z = \min(X_1, \ldots, X_n)$

$N = i$ if $X_i = Z$

$P(N = i,\ Z > z)$

$= P(X_j > X_i > z,\ j \neq i)$

$= \int_0^\infty P(X_j > X_i > z,\ j \neq i \mid X_i = x)\, \lambda_i e^{-\lambda_i x}\, dx$

$$= \int_z^\infty P(X_j > x,\ j \neq i)\ \lambda_i e^{-\lambda_i x}\, dx$$

$$= \int_z^\infty \left(\prod_{j \neq i} P(X_j > x) \right) \lambda_i e^{-\lambda_i x}\, dx$$

$$= \lambda_i \int_z^\infty \left(\prod_{j \neq i} e^{-\lambda_j x} \right) e^{-\lambda_i x}\, dx$$

$$= \lambda_i \int_z^\infty e^{-\lambda x}\, dx, \text{ where } \lambda = \lambda_1 + \dots + \lambda_n$$

$$= \frac{\lambda_i}{\lambda} e^{-\lambda z}$$

So, $P(N = i,\ Z > z) = \frac{\lambda_i}{\lambda} e^{-\lambda z}$ ——— ②

Now, using ②,

$$P(N = i) = \frac{\lambda_i}{\lambda}$$

$$P(Z > z) = e^{-\lambda z}$$

So, (a) N and Z are independent

(b) $Z \sim Exp(\lambda)$

(2) Sum of iid exponentials

$\{X_1, \ldots X_n\}$ iid, $X_i \sim \mathcal{E}xp(\lambda), \forall i$

$Z = X_1 + \cdots + X_n$

LST: $\phi_Z(s) = E\left(e^{-sZ}\right)$

$= E\left(e^{-s(X_1+\cdots+X_n)}\right)$

$\{X_i\}$ indep. $\rightarrow$ $= \prod_{i=1}^{n} E\, e^{-sX_i}$

$$\underline{\phi_Z(s) = \left(\frac{\lambda}{\lambda+s}\right)^n}$$

LST of a Gamma(n,λ) r.v.

a.k.a. Erlang(n,λ)

So, $Z \sim$ Gamma(n,λ)

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(3) Random sum of i.i.d. exponentials

$Z = X_1 + \cdots + X_N$, where

$N \sim$ Geom(p), $X_i \sim \mathcal{E}xp(\lambda)$ $\forall i$ & independent

(i) What is $E(Z)$?

$$E[Z \mid N=n] = \frac{n}{\lambda}$$

$$E[Z \mid N] = \frac{N}{\lambda}$$

$$E[Z] = E[E[Z \mid N]]$$
$$= E\left[\frac{N}{\lambda}\right] = \frac{1}{\lambda} EN$$
$$= \frac{1}{p\lambda}$$

(ii) Distribution of Z:

$$\underline{E[e^{-sZ} \mid N=n] = \left(\frac{\lambda}{\lambda+s}\right)^n}$$

From previous part (See ② above)

$$E[e^{-sZ}] = \sum_{n=1}^{\infty} E[e^{-sZ} \mid N=n]\, P(N=n)$$
$$= \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda+s}\right)^n (1-p)^{n-1} p$$
$$= \frac{\lambda p}{\lambda+s} \sum_{n=0}^{\infty} \left(\frac{\lambda(1-p)}{s+\lambda}\right)^n$$
$$= \frac{\lambda p}{\lambda+s} \times \frac{1}{1-\left(\frac{\lambda(1-p)}{s+\lambda}\right)}$$

$$\phi_Z(s) = \frac{\lambda p}{s+\lambda p}$$

So, $Z \sim \mathcal{E}xp(\lambda p)$

Lecture 2

Poisson processes

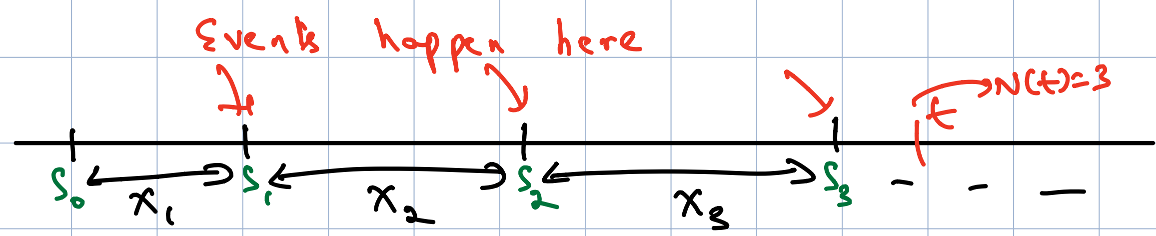
Used to model "counting events one at a time"

e.g. ① # births in a hospital

② # arrivals at a service system

③ # calls made

Let $\{X_n, n \geq 1\}$ be a sequence of non-negative r.v.s representing inter-event times



$S_0 = 0, \quad \underbrace{S_n = X_1 + X_2 + \cdots + X_n}_{\text{time of occurence of nth event}}, \; n \geq 1$

For $t \geq 0$, define $\underbrace{N(t) = \max\{n \geq 0 \mid S_n \leq t\}}_{\text{\# events that occured in } [0,t]}$

$\{N(t), t \geq 0\} \leftarrow$ counting process

Def: $\{N(t), t\geq 0\}$ generated by $\{X_n, n\geq 1\}$ is a Poisson process with parameter $\lambda>0$, if $\{X_n, n\geq 1\}$ is a sequence of i.i.d. $Exp(\lambda)$ r.v.s

We shall use $PP(\lambda)$ to denote this process.

$N(0)=0$ & each event causes a jump of 1.

Theorem: Let $\{N(t), t\geq 0\}$ be $PP(\lambda)$

Then,

$$P(N(t)=k) = \underline{\frac{e^{-\lambda t}(\lambda t)^k}{k!}}, \quad k=0,1,2,\ldots$$

Poisson distribution with parameter λt

Pf: $N(t)\geq k \iff$ k or more events occured in $[0,t]$

$\iff$ k th event occured at/before t

$N(t)\geq k \iff S_k \leq t$

$P(N(t)\geq k) = P(S_k\leq t)$

S_k = sum of "k" i.i.d $Exp(\lambda)$

$$S_k \sim \text{Gamma}(k, \lambda)$$

$$P(S_k \le t) = \begin{cases} 1 - e^{-\lambda t} \displaystyle\sum_{r=0}^{k-1} \frac{(\lambda t)^r}{r!}, & \text{if } t \geq 0 \\ 0, & \text{else} \end{cases}$$

$$\begin{aligned} P(N(t) = k) &= P(N(t) \geq k) - P(N(t) \geq k+1) \\ &= P(S_k \le t) - P(S_{k+1} \le t) \\ &= \left(1 - e^{-\lambda t} \sum_{r=0}^{k-1} \frac{(\lambda t)^r}{r!}\right) - \left(1 - e^{-\lambda t} \sum_{r=0}^{k} \frac{(\lambda t)^r}{r!}\right) \\ &= \frac{e^{-\lambda t} (\lambda t)^k}{k!} \end{aligned}$$

$\blacksquare$

Example Suppose customers arrive at a post office according to PP with rate 10/hour.

(i) What is the distribution of # arrivals during an 8-hour day?

$$N(8) \sim \text{Poisson}(10 \cdot 8) = \text{Poiss}(80)$$

$$P(N(8) = k) = \frac{e^{-80} (80)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

(ii) What is expected # of customers during an 8-hour day?

$$E(N(8)) = 80.$$

Shifted Poisson process

Let $\{N(t), t \geq 0\}$ be $PP(\lambda)$

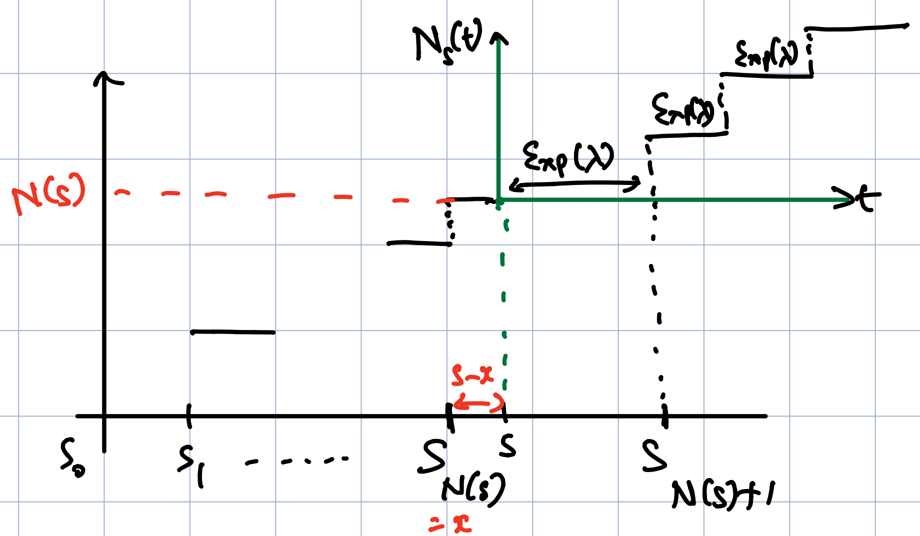
Fix $s \geq 0$,

$$N_s(t) = N(t+s) - N(s), \quad t \geq 0$$

The process $\{N_s(t), t \geq 0\}$ is shifted PP

Theorem: Shifted PP $\{N_s(t), t \geq 0\}$ is $PP(\lambda)$ and is independent of $\{N(u), 0 \leq u \leq s\}$

Pf:



$$P\left(S_{N(s)+1} - s > y \,\middle|\, N(s) = k,\ S_{N(s)} = x,\ N(u): 0 \leq u \leq s\right)$$

$$= P\left(S_k + X_{k+1} - s > y \,\middle|\, N(s) = k,\ S_k = x,\ N(u): 0 \leq u \leq s\right)$$

$$= P\left(X_{k+1} > s + y - x \,\middle|\, X_{k+1} > s - x\right)$$

$$= e^{-\lambda y}.$$

So, the first event in $\{N_s(t), t \geq 0\}$ occurs after an $Exp(\lambda)$ time.

The rest of inter-event times are $Exp(\lambda)$

So, $\{N_s(t), t \geq 0\}$ is $PP(\lambda)$. ■

Stationary and independent increments:

Let $\{X(t), t \geq 0\}$ be a continuous real-valued stochastic process. For given $s, t \geq 0$,

$X(s+t) - X(s) \leftarrow$ increment over $(s, s+t]$

$\{X(t), t \geq 0\}$ is said to have stationary & independent increments if

(i) the distribution of the increment over $(s, s+t]$ is independent of s

(ii) the increments over non-overlapping intervals are independent

<u>Theorem</u>: A Poisson process has stationary & independent increments.

<u>Pf</u>: Let $\{N(t), t \geq 0\}$ be $PP(\lambda)$

We know $N_s(t) = N(t+s) - N(s)$

$\{N_s(t), t \geq 0\}$ is $PP(\lambda)$

So, $P(N(t+s)-N(s)=k) = \underbrace{\frac{e^{-\lambda t}(\lambda t)^k}{k!}}_{\text{independent of } s}$, $k=0,1,2,\ldots$

Hence, $\{N(t), t\geq 0\}$ has stationary increments

Next, let $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$.

$\underline{N(t_2)-N(t_1)}$ and $\underline{N(t_4)-N(t_3)}$

increments over non-overlapping intervals $(t_1, t_2]$, $(t_3, t_4]$

From Shifted-PP theorem,

$$N_{t_3}(t_4-t_3) = N(t_4)-N(t_3) \text{ is}$$

$$\text{independent of } \{N(u) : 0 \leq u \leq t_3\}$$

i.e., $N(t_4)-N(t_3)$ is independent of $N(t_2)-N(t_1)$

Hence, $\{N(t), t\geq 0\}$ has independent increments. ■

Lecture-3

Finite dimensional distributions

Let $\{N(t), t\geq 0\}$ be $PP(\lambda)$

$\underbrace{0 \leq t_1 \leq t_2 \leq \cdots \leq t_n}_{\text{real numbers}}$ & $\underbrace{0 \leq k_1 \leq \cdots \leq k_n}_{\text{integers}}$

$$P(N(t_1)=k_1,\ N(t_2)=k_2,\ \ldots,\ N(t_n)=k_n)$$

$$= e^{-\lambda t_n}\frac{(\lambda t_1)^{k_1}}{k_1!}\frac{(\lambda(t_2-t_1))^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{(\lambda(t_n-t_{n-1}))^{k_n-k_{n-1}}}{(k_n-k_{n-1})!}$$

Pf: $P(N(t_1)=k_1,\ N(t_2)=k_2,\ \ldots,\ N(t_n)=k_n)$

$$= P(N(t_1)=k_1,\ N(t_2)-N(t_1)=k_2-k_1,\ \ldots,\ N(t_n)-N(t_{n-1})=k_n-k_{n-1})$$

indep. increments

$$= P(N(t_1)=k_1)\,P(N(t_2)-N(t_1)=k_2-k_1)\cdots P(N(t_n)-N(t_{n-1})=k_n-k_{n-1})$$

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{k_1}}{k_1!}\ \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{e^{-\lambda(t_n-t_{n-1})}(\lambda(t_n-t_{n-1}))^{k_n-k_{n-1}}}{(k_n-k_{n-1})!}$$

$$= e^{-\lambda t_n}\frac{(\lambda t_1)^{k_1}}{k_1!}\frac{(\lambda(t_2-t_1))^{k_2-k_1}}{(k_2-k_1)!}\cdots\frac{(\lambda(t_n-t_{n-1}))^{k_n-k_{n-1}}}{(k_n-k_{n-1})!}$$

$\blacksquare$

EXAMPLE:

① Auto-covariance function

Let $\{N(t), t \geq 0\}$ be $PP(\lambda)$

What is $Cov(N(s), N(s+t))$, $s, t \geq 0$?

$$E(N(s)\,N(s+t))$$
$$= E[N(s)(N(s+t) - N(s) + N(s))]$$
$$= E[N(s)(N(s+t) - N(s))] + E(N(s)^2)$$
$$= E(N(s))\,E(N(s+t) - N(s)) + E(N(s)^2)$$
$$= \lambda s\,\lambda t + \lambda s + (\lambda s)^2$$

$Var(N(s)) = \lambda s$

$E(N(s)^2) - (E(N(s)))^2 = \lambda s$

$E(N(s)^2) = \lambda s + (\lambda s)^2$

$$Cov(N(s), N(s+t)) = E(N(s)\,N(s+t)) - E\,N(s)\,E\,N(s+t)$$
$$= \lambda s\,\lambda t + \lambda s + (\lambda s)^2 - \lambda s\,\lambda(s+t)$$
$$= \lambda s \longrightarrow \text{does not depend on "}t\text{"}$$

② Recall the post-office example

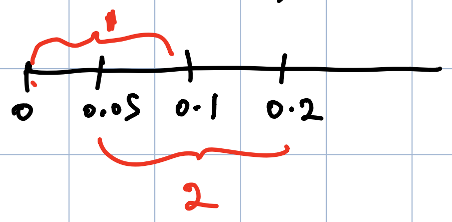
Customers arrive according to PP with rate 10/hour

$(0, 0.1)$

$P($ one customer arrives between 1pm & 1:06pm, and two customers arrive between 1:03pm & 1:12pm$)$

$(0,0.2) - (0,0.05)$

$$= P\left(N(0.1) = 1,\ N(0.2) - N(0.05) = 2 \right)$$



$$= \sum_{k=0,1} P\left(N(0.05) = k,\ N(0.1) - N(0.05) = 1-k,\ N(0.2) - N(0.1) = 1+k \right)$$

$$= \sum_{k=0,1} P(N(0.05) = k)\, P(N(0.1) - N(0.05) = 1-k) \times P(N(0.2) - N(0.1) = 1+k)$$

$$= \sum_{k=0,1} \frac{e^{-0.5}(0.5)^k}{k!} \frac{e^{-0.5}(0.5)^{1-k}}{(1-k)!} \frac{e^{-1}(1)^{1+k}}{(1+k)!}$$

$$= 0.1015$$

Alternate characterization of Poisson process

Thm: $\{N(t), t \geq 0\}$ is $PP(\lambda)$ ie., inter event times iid $Exp(\lambda)$

"done" $\Downarrow$ $\Uparrow$ "homework"

(i) $\{N(t), t \geq 0\}$ has stationary & independent increments

(ii) $N(t) \sim \text{Poisson}(\lambda t),\ \forall t \geq 0$

"So (i) & (ii) together can be used to define PP(λ)"

Event times in a Poisson process

$$N(t)=n$$

$$
\begin{array}{cccccccc}
0 & S_1 & S_2 & - & - & - - - & S_n & t \\
 & \tilde{U}_1 & \tilde{U}_2 & & & & \tilde{U}_n &
\end{array}
$$

$(\tilde{U}_1, \ldots, \tilde{U}_n)$ = order statistics (see below)

Given $N(t)=n$, what is the distribution of $(S_1, \ldots, S_n)$

Let $(U_1, \ldots, U_n)$ be n iid Unif $(0,t)$ r.v.s

Let $\tilde{U}_1, \ldots, \tilde{U}_n$ be the order statistics of $(U_1, \ldots U_n)$.

$$\tilde{U}_1 \leq \tilde{U}_2 \leq \cdots \leq \tilde{U}_n$$

In particular,

$$\tilde{U}_1 = \min(U_1, \ldots U_n)$$

Unif(0,t)

$$\tilde{U}_n = \max(U_1, \ldots U_n)$$

$$\tilde{U}_i = i\text{th smallest among } (U_1, \ldots U_n)$$

$$\underbrace{(S_1, \ldots S_n)}_{\text{distribution of event times}} \sim \underbrace{(\tilde{U}_1, \ldots, \tilde{U}_n)}_{\text{distribution of order statistics of } n \text{ unif r.v.s } (0,t)}$$

given $N(t)=n$

$\equiv$

Lecture-4

Distribution of order statistics

$X_1, \ldots X_n$ i.i.d. continuous r.v.s with CDF F & density f

$X_{(1)} = \min(X_1, \ldots X_n)$, $X_{(n)} = \max(X_1, \ldots X_n)$

$X_{(i)} =$ ith smallest among $X_1, \ldots X_n$

$$P(X_{(i)} \le x) = \sum_{k \ge i} \binom{n}{k} (F(x))^k (1-F(x))^{n-k}$$

Differentiating,

$$f_{(i)}(x) = f(x) \sum_{k\ge i} \binom{n}{k} k (F(x))^{k-1} (1-F(x))^{n-k}$$

$$- f(x) \sum_{k \ge i} \binom{n}{k} (n-k) (F(x))^k (1-F(x))^{n-k-1}$$

$$= f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!(k-1)!} (F(x))^{k-1} (1-F(x))^{n-k}$$

$$- f(x) \sum_{k=i}^{n-1} \frac{n!}{k!\,(n-k-1)!} (F(x))^k (1-F(x))^{n-k-1}$$

$k<n$

$$= f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!(k-1)!} (F(x))^{k-1} (1-F(x))^{n-k}$$

$$- f(x) \sum_{j=i+1}^{n} \frac{n!}{(n-j)!\,(j-1)!} (F(x))^{j-1} (1-F(x))^{n-j}$$

$$f_{(i)}(x) = \frac{n!}{(n-i)!\,(i-1)!}\, f(x)\, \left(F(x)\right)^{i-1} \left(1-F(x)\right)^{n-i}$$

For $X_{(i)} = x$,

you need $(i-1)$ of $X_1, \dots X_n$ $\le x$

$(n-i)$ —"— $> x$

& one of them $= x$

# of partitions of n into 3 groups containing "$i-1$", "$n-i$" & "1" members

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Let $U_1, \dots U_n$ be n iid Unif$(0,t)$ r.v.s

$\tilde{U}_1, \dots, \tilde{U}_n$ denote the order statistics.

Then the density $f_{(i)}$ of $\tilde{U}_i$ is given by

$$f_{(i)}(u) = \frac{n!}{(n-i)!(i-1)!}\, \frac{1}{t} \left(\frac{u}{t}\right)^{i-1} \left(1-\frac{u}{t}\right)^{n-i}$$

Beta density (H.W. read about Beta r.v.s)

The joint density $f(\cdot, \dots \cdot)$ of $(\tilde{U}_1, \dots \tilde{U}_n)$ is

$$(*) \rightarrow f(t_1, \dots, t_n) = \begin{cases} n!/t^n & \text{if } t_1 \le t_2 \le \dots \le t_n \\ 0 & \text{else} \end{cases}$$

"Do your own proof of (*)"

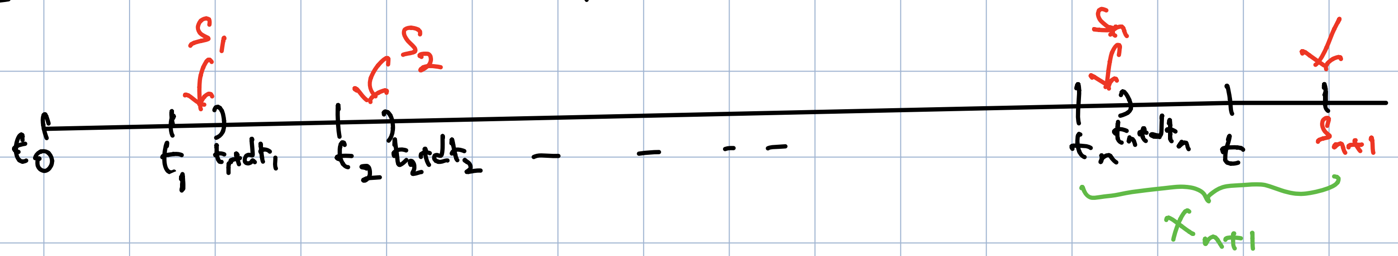
Campbell theorem $\{N(t), t \geq 0\}$ is $PP(\lambda)$

Given $N(t) = n$,

$(S_1, \ldots, S_n) \sim (\tilde{U}_1, \ldots, \tilde{U}_n)$

has the same distribution

Pf: Fix $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_n \leq t$ $\quad S_{n+1} > t$



$$P\left(S_i \in (t_i, t_i + dt_i), \text{ for } i = 1 \ldots n \,\middle|\, N(t) = n\right)$$

$$= \frac{P\left(S_i \in (t_i, t_i + dt_i), \text{ for } i = 1 \ldots n, \ N(t) = n\right)}{P(N(t) = n)}$$

$$= \frac{P\left(S_i \in (t_i, t_i + dt_i), \text{ for } i = 1 \ldots n, \ S_{n+1} > t\right)}{P(N(t) = n)}$$

$$= \frac{P\left(X_i \in (t_i - t_{i-1}, t_i - t_{i-1} + dt_i), i = 1 \ldots n, \ X_{n+1} > t - t_n\right)}{P(N(t) = n)}$$

$$= \prod_{i=1}^{n} \left(\lambda e^{-\lambda(t_i - t_{i-1})} dt_i\right) e^{-\lambda(t - t_n)} \Big/ \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= \frac{e^{-\lambda t}\lambda^n}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}\, dt_1 \cdots dt_n = \frac{n!}{t^n}\, dt_1 \cdots dt_n$$

So, the density of $(S_1, \ldots, S_n)$ given $N(t)=n$ is

$$f(t_1, \ldots, t_n) = \begin{cases} \dfrac{n!}{t^n} & \text{if } 0 \le t_1 \le \cdots \le t_n \le t \\ 0 & \text{else} \end{cases}$$

i.e., f is the density of $(\tilde{U}_1, \ldots, \tilde{U}_n)$. $\blacksquare$

EXAMPLE:

① $\{N(t), t \geq 0\}$ $PP(\lambda)$

$$P(S_1 > s \mid N(t) = n)$$

Density of S_1: given $N(t)=n$

$$f_{(1)}(u) = \frac{n}{t}\left(1 - \frac{u}{t}\right)^{n-1}, \quad 0 \le u \le t$$

$$P(S_1 > s \mid N(t)=n) = \int_s^t f_{(1)}(u)\, du$$

$$= \int_s^t \frac{n}{t}\left(1-\frac{u}{t}\right)^{n-1} du$$

$$= \left(1 - \frac{s}{t}\right)^n$$

$$E(S_{(i)} \mid N(t)=n) = E(\tilde{U}_i)$$

H.w. $\tilde{U}_i = \text{Beta}(?,?)$

expectation of a Beta r.v. (read)

$$= \frac{it}{n+1}$$

$$E(S_{n+1} \mid N(t)=n) = t + 1/\lambda$$

n events happened here

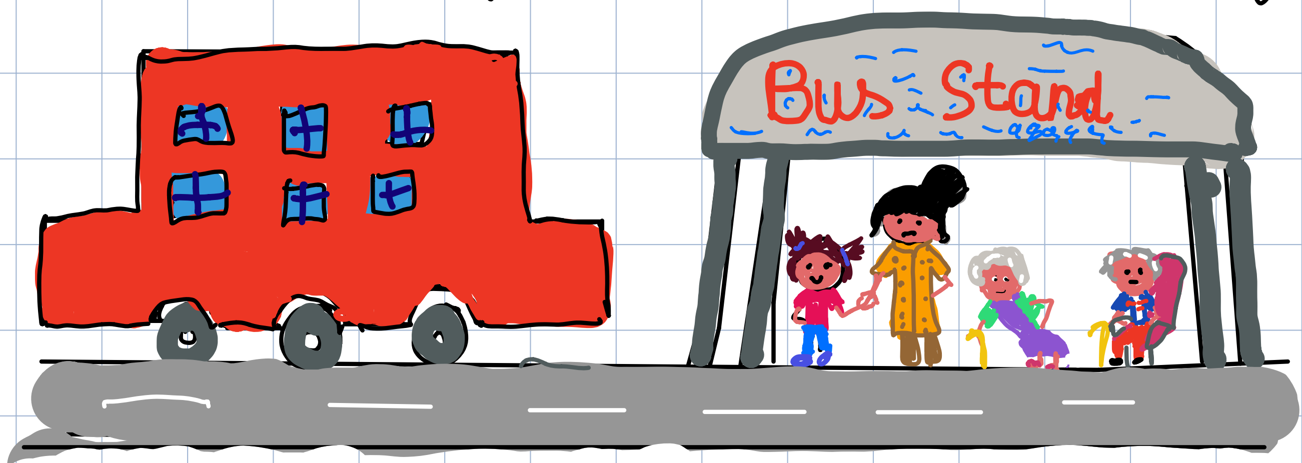
$\text{Exp}(\lambda)$

0   t S_{n+1}

$$E(S_{n+2} \mid N(t)=n) = t + \frac{2}{\lambda} \quad \text{\& so on}$$

---

(2) Passengers arrive at a bus stand according to $PP(\lambda)$. Buses leave every "T" time units. Bus is "large" enough to take all waiting passengers. What is the avg. waiting time of the passengers?



Soln

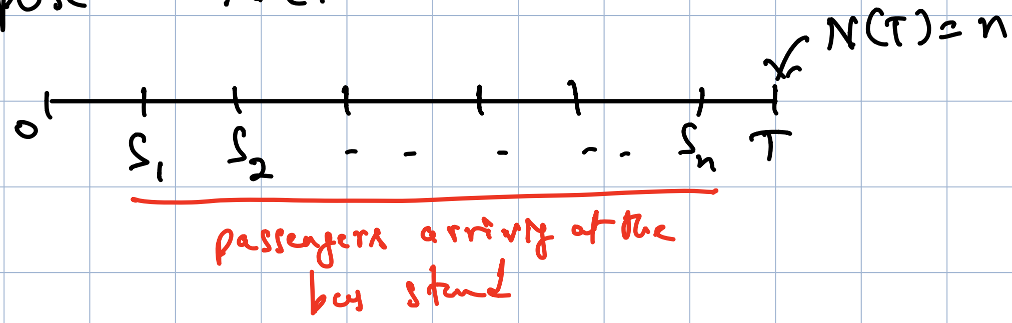
Sol^n $\{N(t), t \geq 0\}$ $PP(\lambda)$

Time 0: Suppose a bus has left. & the bus stand is empty at time 0

Analyze the waiting time in "$[0, T]$"

waiting passengers at T is $N(T)$

Suppose $N(T) = n > 0$



ith passenger waits for $T - S_i$

Avg waiting time (r.v.) $= \frac{1}{n} \sum_{i=1}^{n} (T - S_i)$

$$= T - \frac{1}{n} \sum_{i=1}^{n} S_i$$

Given $N(t) = n$, $(S_1, \dots, S_n) \sim (\tilde{U}_1, \dots, \tilde{U}_n)$

$$\sum_{i=1}^{n} S_i = \sum_{i=1}^{n} \tilde{U}_i = \sum_{i=1}^{n} U_i \sim \text{Unif}(0,T]$$

So, $E(\text{Avg waiting time} \mid N(t)=n)$

$$= T - \frac{1}{n} E\left(\sum_{i=1}^{n} U_i\right)$$

$$= T - \frac{T}{2}$$

$$= \frac{T}{2}.$$

[redacted]

Lecture - 5

Superposition of Poisson processes.

(or) Combining Poisson processes.

Let $\{N_i(t), t \geq 0\}$, $i = 1 \dots r$ be independent Poisson processes with parameters $\lambda_1, \dots \lambda_r$

$$N(t) = N_1(t) + \dots + N_r(t), \quad t \geq 0$$

<u>Example!</u> Telephone exchange

Calls arrive from 2 sources

domestic   international

Each Source has Poisson arrivals & the rates vary

<u>Theorem</u>: $\{N_i(t), t \geq 0\}$ $i = 1 \dots r$ independent PP with rates $\lambda_1, \dots \lambda_r$

$$N(t) = N_1(t) + \dots + N_r(t), \quad t \geq 0$$

$\{N(t), t \geq 0\}$ is $PP(\lambda)$, where $\lambda = \lambda_1 + \dots + \lambda_r$

<u>Pf</u>: Enough to show

(i) $\{N(t), t \geq 0\}$ has stationary & indep. increments

(ii) $N(t) \sim \text{Poisson}(\lambda t)$

(i) follows easily since $\{N_i(t), t \geq 0\}$ has stationary & indep. increments

(ii) Fix t

$N_i(t) \sim \text{Poisson}(\lambda_i t)$, $i = 1 \ldots r$

$N(t) = N_1(t) + \cdots + N_r(t)$, (indep. sum)

$N(t) \sim \text{Poisson}(\lambda t)$

$\blacksquare$

H.w. $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$, X, Y independent

S.T. $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$

Example! Jobs arrive at a computer in four "priority" classes, say 1, 2, 3, 4. Inter-arrival times for jobs from class "i" is Exponential with mean m_i minutes. Here $m_1 = 10$, $m_2 = 15$, $m_3 = 30$, $m_4 = 60$. Suppose the arrival streams are independent.

$N(t)$ = total # of jobs across classes in $(0, t]$

Characterize $N(t)$.

$N_i(t) \sim PP\left(\frac{60}{m_i}\right)$, $i = 1 \ldots 4$

$N(t) \sim PP(\lambda)$, where

$\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \frac{60}{10} + \frac{60}{15} + \frac{60}{30} + \frac{60}{60} = 13/\text{hour}$

$\{N_i(t), t \geq 0\}$ is $PP(\lambda_i)$, indep & $N(t) = \sum_{i=1}^{r} N_i(t)$

$Z_n = i$ if nth event of $\{N(t), t \geq 0\}$ is of type "i".

Claim: $$P(Z_n = i) = \frac{\lambda_i}{\lambda}, \quad i = 1 \dots r$$

and $\{Z_n, n \geq 1\}$ is iid.

Pf: Let S_i be the time of occurence of the first event in $\{N_i(t), t \geq 0\}$

$$S_i \sim Exp(\lambda_i), \quad i = 1 \dots r$$

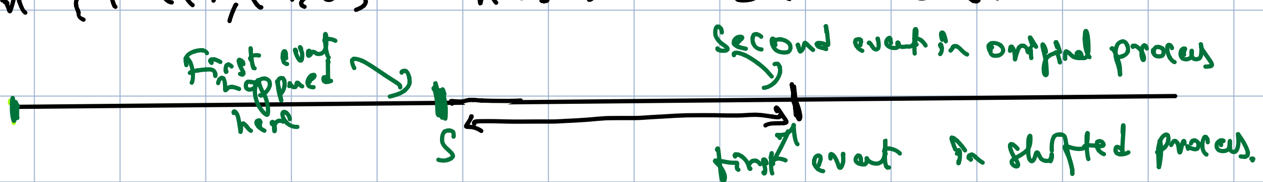
$$P(Z_1 = i) = P\left(S_i = \min(S_j, j = 1 \dots r)\right) = \frac{\lambda_i}{\lambda}$$

Now, suppose the first event in the combined process occurs at "s".

Consider shifted process $\{\tilde{N}_i(t) = N_i(t+s) - N_i(s), t \geq 0\}$, $i = 1 \dots r$.

$$\tilde{N}_i(t) \sim PP(\lambda_i)$$

So, Z_2, which is the time of occurence of 2^{nd} event in $\{N(t), t \geq 0\}$ has the same distribution as Z_1



and, Z_2 is independent of Z_1

& so on.

Hence $\{Z_n\}$ indep & $P(Z_n = i) = \frac{\lambda_i}{\lambda}$ ■

Example:

Customers arrive at a bank in 3 categories.

Category 1: deposit money → takes 3 min

Category 2: withdraw money → takes 4 min

Category 3: steal money → takes 6 min

Category 'i' arrivals $\sim PP(\lambda_i)$

$\lambda_1 = 20, \quad \lambda_2 = 15, \quad \lambda_3 = 5$

What is the avg. transaction time?

$Z \leftarrow$ category of a typical customer

$$P(Z=1) = \frac{20}{40}, \quad P(Z=2) = \frac{15}{40}, \quad P(Z=3) = \frac{5}{40}$$

$$\text{Avg time} = \frac{20}{40} \times 3 + \frac{15}{40} \times 4 + \frac{5}{40} \times 6 = \frac{150}{40} = 3.75$$

Splitting a Poisson process

Let $\{N(t),\ t \geq 0\}$ be $PP(\lambda)$

Take each event & classify it as

type 1 w.p. p
type 2 w.p. $q = 1-p$

Bernoulli splitting

$N_1(t)$ = # events in $(0,t]$ of type 1
$N_2(t)$ = # events in $(0,t]$ of type 2

$$N(t) = N_1(t) + N_2(t),\quad t \geq 0$$

<u>Claim</u>: $\{N_1(t), t \geq 0\} \sim PP(\lambda p)$ &
$\{N_2(t), t \geq 0\} \sim PP(\lambda q)$

& they are independent.

Think about this:-
Toss a coin with bias p "N" times. N is deterministic

N_1 = # heads $\qquad N_2$ = # tails

N_1 & N_2 are NOT independent

Suppose $N \sim \text{Poisson}(\lambda)$

In this case, N_1 & N_2 will be independent

Pf:- We will show $\{N_1(t), t \geq 0\} \sim PP(\lambda p)$

Let $\{X_n, n \geq 1\}$ be inter-arrival times in the original process $\{N(t), t \geq 0\}$

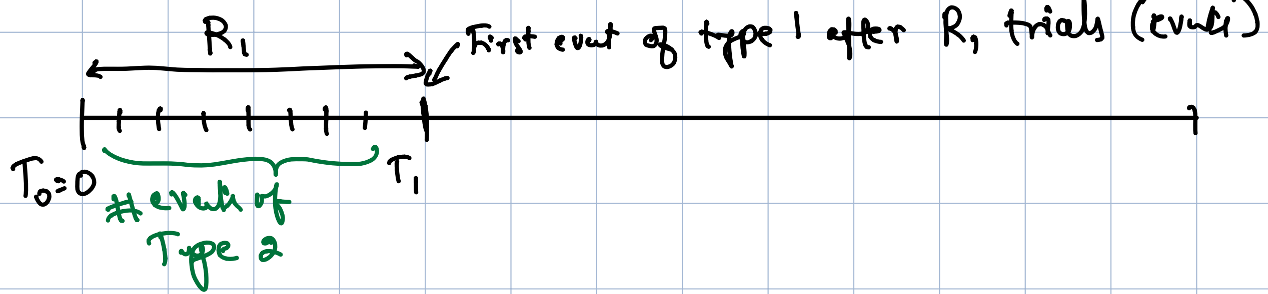
$\{X_n, n \geq 1\} \sim$ iid $Exp(\lambda)$

Let $\{Y_n, n \geq 1\}$ be the inter-arrival times in $\{N_1(t), t \geq 0\}$

Enough to show that $\{Y_n, n \geq 1\} \sim$ iid $Exp(\lambda p)$

Let $R_1 \sim$ Geometric (p), i.e.,

$$P(R_1 = k) = (1-p)^{k-1} p, \quad k \geq 1$$



$$Y_1 = \sum_{i=1}^{R_1} X_i$$

From way above, $Y_1 \sim Exp(\lambda p)$

To proceed further,

Let $\{R_n, n \geq 1\}$ be iid Geometric(p) r.v.s

$T_0 = 0$, $T_n = R_1 + R_2 + R_3 + \cdots + R_n$, $n \geq 1$

$$Y_n = \sum_{i=T_{n-1}+1}^{T_n} X_i, \quad n \geq 1$$

Easy to see that $\{Y_n, n \geq 1\}$ is iid.

& since $Y_1 \sim Exp(\lambda p)$, we have

$\{N_1(t), t \geq 0\}$ is $PP(\lambda p)$

A similar argument works for showing

$\{N_2(t), t \geq 0\}$ is $PP(\lambda q)$, $q = 1-p$.

"Establishing independence":

Enough to show:

$$P(N_1(t) = i, N_2(t) = j) = P(N_1(t) = i)\, P(N_2(t) = j) \quad \forall i, j \geq 0$$

<u>Pf:</u> $P(N_1(t) = i, N_2(t) = j)$

$$= P(N_1(t) = i, N_2(t) = j \mid N(t) = i+j)\, P(N(t) = i+j)$$

$$= \binom{i+j}{i} p^i q^j \times \frac{e^{-\lambda t} (\lambda t)^{i+j}}{(i+j)!}$$

$$= \left(\frac{e^{-\lambda p t} (\lambda p t)^i}{i!} \right) \left(\frac{e^{-\lambda q t} (\lambda q t)^j}{j!} \right)$$

$$= P(N_1(t) = i) \; P(N_2(t) = j) \qquad \blacksquare$$

Lecture - 6

H.W. Let $\{N(t), t \geq 0\}$ be $PP(\lambda)$

Consider a r-way split using

$p_1 \ldots p_r$, $\quad p_i \geq 0$, $\quad \sum_{i=1}^{r} p_i = 1$

Each event is classified as type "i" w.p. p_i, $i = 1 \sim r$

Let $N_i(t)$ = # events of type "i" in $(0, t]$

Then, Show that

$\{N_i(t), t \geq 0\}$ is a Poisson process with rate λp_i,

for $i = 1 \ldots r$

Also, show that the "r" processes $\{N_i(t), t \geq 0\}$ $i = 1 \ldots r$ are independent

---

EXAMPLE: Geiger counter → counts radioactive particles (GC)

Suppose particles arrive at GC according to $PP(1000)$ per second

The counter fails to count a particle w.p. 0.1, independent of everything else

Suppose the counter registers "4" particles in 0.01 secs.

Find Prob(at least 6 particles arrived at the counter in 0.01 secs)

<u>Soln</u> $N(t) \to$ # particles that arrive
$N_1(t) \to$ # particles GC counted
$N_2(t) \to$ # particles GC missed
} all in $(0,t]$

$\{N(t), t \geq 0\}$ is PP(1000)
$\{N_1(t), t \geq 0\}$ is PP(900)
$\{N_2(t), t \geq 0\}$ is PP(100)
$N_1(t)$ indep. of $N_2(t)$

$$P(N(0.01) \geq 6 \mid N_1(0.01) = 4)$$
$$= P(N_2(0.01) \geq 2 \mid N_1(0.01) = 4)$$
$$= P(N_2(0.01) \geq 2) \qquad N_2(0.01) \sim \text{Poisson}(1)$$
$$= 0.264$$

using

Non-homogeneous Poisson process

Homogeneous PP $\to$ (i) $N(t) \sim \text{Poisson}(\lambda t)$
(ii) stationary & indep. increments

Non-Homogeneous PP $\to$ (i) $N(t) \sim \text{Poisson}(\cancel{\lambda t})$ depends on the interval
(ii) ~~stationary~~ & indep. increments

Let $\lambda : [0,\infty) \to [0,\infty)$ be a given function.

$$\Lambda(t) = \int_0^t \lambda(s)\, ds \quad \leftarrow \text{Assume } \lambda \text{ is integrable}$$

Non-homogeneous PP:

$\{N(t), t \geq 0\}$ non-homogeneous PP if

(i) $\{N(t), t \geq 0\}$ has indep. increments

(ii) $N(t) \sim \text{Poisson}(\Lambda(t))$, $t \geq 0$

Remark: ① $\lambda(s) \equiv \lambda$, $\forall s$ then $\Lambda(t) = \lambda t$ &

we recover the homogeneous case

② (i) $N(0) = 0$

(ii) $N(t+s) - N(s) \underset{\text{H.W.}}{\sim} P(\Lambda(t+s) - \Lambda(s))$

Finite-dimensional distributions:

Let $\{N(t), t \geq 0\}$ be $NPP(\lambda(\cdot))$.

Fix $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ &

$0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$

$$P(N(t_1)=k_1,\ N(t_2)=k_2,\ \ldots,\ N(t_n)=k_n)$$

$$= e^{-\Lambda(t_n)} \frac{(\Lambda(t_1))^{k_1}}{k_1!} \cdot \frac{(\Lambda(t_2)-\Lambda(t_1))^{k_2-k_1}}{(k_2-k_1)!} \cdots \times \frac{(\Lambda(t_n)-\Lambda(t_{n-1}))^{k_n-k_{n-1}}}{(k_n-k_{n-1})!}$$

Pf: Follows by a parallel argument to the proof in homogeneous case. ∎

Reading exercise: Check Example 5.18 in the textbook.

Event times in NPP:

S_n = time of occurrence of nth event

$X_n = S_n - S_{n-1}$

$$P(X_1 > t) = P(N(t)=0) = e^{-\Lambda(t)}$$

For $n \geq 1$,

$$P(X_{n+1} > t) = \int_0^\infty \lambda(s)\, e^{-\Lambda(t+s)} \frac{\Lambda(s)^{n-1}}{(n-1)!}\, ds$$

H.W.

Hint: (i) Find $P(X_{n+1} > t \mid S_n = s)$

(ii) $P(S_n \le s) = P(N(s) \ge n) =$ Fill here

CDF

$\Big\downarrow$ differentiate

density $f_{S_n}(s)$

(iii) $P(X_{n+1} > t)$

$$= \int_0^\infty P(X_{n+1} > t \mid S_n = s)\, f_{S_n}(s)\, ds$$

Event times:

Fix $t \ge 0$

Let U_i, $i = 1 \dots n$ be "n" iid r.v.s such that

$$P(U_i \le s) = \frac{\Lambda(s)}{\Lambda(t)}, \quad 0 \le s \le t$$

Let $\tilde{U}_1, \dots \tilde{U}_n$ denote the order statistics of $U_1 \dots U_n$

Then,

$$(S_1, \dots, S_n) \sim (\tilde{U}_1, \dots, \tilde{U}_n)$$

H.W. Prove the above claim.

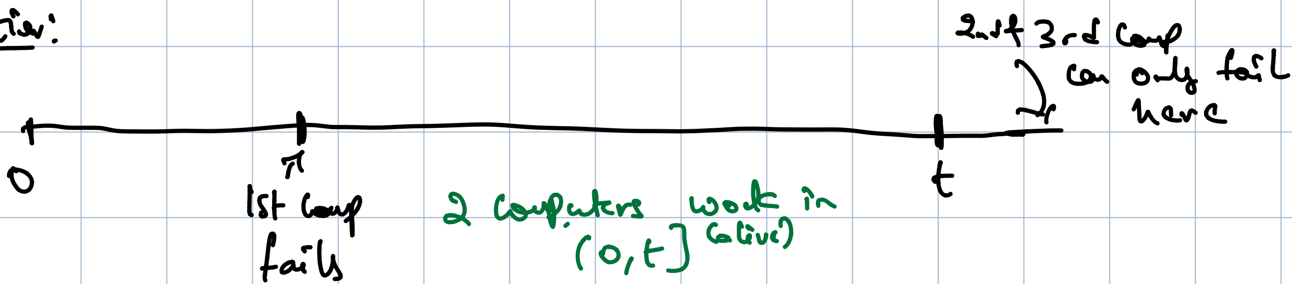
Hint: Follow the proof for the homogeneous case.

Lecture-7

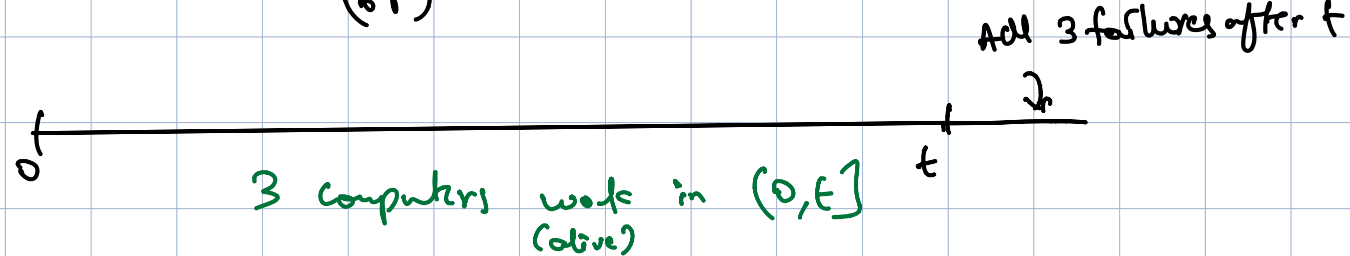
A few problems to solve

S.6 A spaceship is controlled by 3 indep. computers. The ship can function as long as at least 2 of these computers work. Suppose the lifetimes of these computers are iid $\mathcal{E}xp(\lambda)$. Assume all 3 computers are working at $t=0$. Prob(ship working over $(0,t]$)?

Solution:



(or)



Let X_1, X_2, X_3 be lifetimes of comp 1, 2, 3.

$X_{(1)}, X_{(2)}, X_{(3)} \to$ order statistics of (X_1, X_2, X_3)

$$P(X_{(2)} \le t) = \sum_{k \ge 2} \binom{3}{k} (F(t))^k (1 - F(t))^{3-k}$$

$$= 3(1-e^{-\lambda t})^2 e^{-\lambda t} + (1-e^{-\lambda t})^3$$

$$P(X_{(2)} > t) = 1 - \left(3(1-e^{-\lambda t})^2 e^{-\lambda t} + (1-e^{-\lambda t})^3\right)$$

Alternately,

$P(X_{(2)} > t)$

$= (e^{-\lambda t})^3 + 3(1- e^{-\lambda t} e^{-2\lambda t})$

5.15 Suppose $U \sim Unif(0,1)$

Show that $-\ln(U) \sim Exp(1)$

$P(-\ln(U) > x)$

$= P(U < e^{-x})$

$= e^{-x}$

So, $-\ln(U)$ is $Exp(1)$

Remark: To generate an $Exp(1)$ r.v.,
we could generate a unif rand number in $(0,1)$
& use $-\ln(U)$

This is the inverse transform method for simulation of distributions

---

5.26) Let $\{N(t), t \geq 0\}$ be $PP(\lambda)$

We "Bernoulli" split this process into $\{N_1(t), t \geq 0\}$ & $\{N_2(t), t \geq 0\}$, with underlying probabilities p & $(1-p)$ used to classify events as type 1 & 2.

$T_1 \to$ time of occurence of first event in $\{N_1(t), t \geq 0\}$

$T_2 \to$ time of occurence of first event in $\{N_2(t), t \geq 0\}$

Find joint distribution of (T_1, T_2)

Soln) $\{N_1(t), t \geq 0\} \rightarrow PP(\lambda p)$

$\{N_2(t), t \geq 0\} \rightarrow PP(\lambda q)$, $q = 1-p$.

$N_1(t)$ & $N_2(t)$ are indep.

$$P(T_1 \leq x_1) = 1 - e^{-\lambda p x_1}$$

$$P(T_2 \leq x_2) = 1 - e^{-\lambda q x_2}$$

$$P(T_1 \leq x_1, T_2 \leq x_2)$$

$$= P(T_1 \leq x_1) P(T_2 \leq x_2)$$

$$= (1 - e^{-\lambda p x_1})(1 - e^{-\lambda q x_2})$$