Polar Parameters Estimation for LSQ

The expression for Energy Function in Polar Coordinates can be given as

$$E = \sum_{i=1}^{N} (X_i \cos\theta + Y_i \sin\theta - \rho)^2 \tag{1}$$

Partial derivative of (1) with respect to ρ

$$\frac{\partial E}{\partial \rho} = \sum_{i=1}^{N} (X_i \cos\theta + Y_i \sin\theta - \rho) = 0 \tag{2}$$

$$\sum_{i=1}^{N} (X_i \cos\theta + Y_i \sin\theta) - N\rho = 0$$

Rearranging, we have

$$\rho = \frac{1}{N} \sum_{i=1}^{N} (X_i \cos\theta + Y_i \sin\theta)$$

$$\boxed{\rho = \mu_x \cos\theta + \mu_y \sin\theta} \tag{3}$$

Partial derivative of (1) with respect to θ

$$\frac{\partial E}{\partial \theta} = \sum_{i=1}^{N} [(X_i \cos\theta + Y_i \sin\theta - \rho)(Y_i \cos\theta - X_i \sin\theta)] = 0 \tag{4}$$

Expanding (4),

$$\sum_{i=1}^{N} [X_i Y_i \cos^2\theta - X_i^2 \sin\theta\cos\theta + Y_i^2 \sin\theta\cos\theta - X_i Y_i \sin^2\theta] = \rho \sum_{i=1}^{N} [Y_i \cos\theta - X_i \sin\theta]$$

Rearranging,

$$\rho \sum_{i=1}^{N} [Y_i \cos\theta - X_i \sin\theta] = \sum_{i=1}^{N} [X_i Y_i (\cos^2\theta - \sin^2\theta) + (Y_i^2 - X_i^2) \sin\theta\cos\theta]$$

$$\rho \sum_{i=1}^{N} [Y_i \cos\theta - X_i \sin\theta] = \sum_{i=1}^{N} \left[X_i Y_i \cos 2\theta + (Y_i^2 - X_i^2) \frac{\sin 2\theta}{2}\right] \tag{5}$$

Substituting (3) in (5)

$$\left(\mu_x \cos\theta + \mu_y \sin\theta\right) \sum_{i=1}^{N} [Y_i \cos\theta - X_i \sin\theta] = \sum_{i=1}^{N} \left[X_i Y_i \cos 2\theta + (Y_i^2 - X_i^2)\frac{\sin 2\theta}{2}\right]$$

Multiplying on both sides by $\frac{1}{N}$

$$\left(\mu_x \cos\theta + \mu_y \sin\theta\right)\left(\mu_y \cos\theta - \mu_x \sin\theta\right) = E(XY)\cos 2\theta + (E(Y^2) - E(X^2))\frac{\sin 2\theta}{2}$$

Rearranging,

$$\mu_x\mu_y\,(\cos^2\theta - \sin^2\theta) + \left(\mu_y^2 - \mu_x^2\right)\sin\theta\cos\theta = E(XY)\cos 2\theta + (E(Y^2) - E(X^2))\frac{\sin 2\theta}{2}$$

$$\mu_x\mu_y \cos 2\theta + \left(\mu_y^2 - \mu_x^2\right)\frac{\sin 2\theta}{2} = E(XY)\cos 2\theta + (E(Y^2) - E(X^2))\frac{\sin 2\theta}{2}$$

$$\left[(E(X^2) - E(Y^2)) + \left(\mu_y^2 - \mu_x^2\right)\right]\frac{\sin 2\theta}{2} = \left[E(XY) - \mu_x\mu_y\right]\cos 2\theta$$

$$\left[(E(X^2) - \mu_x^2) - \left(E(Y^2) - \mu_y^2\right)\right]\frac{\sin 2\theta}{2} = \left[E(XY) - \mu_x\mu_y\right]\cos 2\theta$$

$$[var(X) - var(Y)]\frac{\sin 2\theta}{2} = [cov(X,Y)]\cos 2\theta$$

$$\boxed{\tan 2\theta = \frac{2\,cov(X,Y)}{[var(X) - var(Y)]}} \quad (6)$$

Thus, the final solutions to θ can be given as follows,

$$\boxed{\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\,cov(X,Y)}{[var(X) - var(Y)]}\right)} \quad (7)$$