Complexity

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Algorithms

- We looked at program correctness.
- For the same problem, there could be multiple algorithms.
- An algorithm is a clearly specified sequence of simple instructions that solve a given problem.
  - An algorithm, by definition, terminates.
  - Otherwise, the sequence of instructions constitutes a procedure.
- The algorithm should be so clear to you that you should be able to make a machine understand it.
  - This is called programming.

Algorithm Efficiency

- For the same problem, there could be multiple algorithms.
- We prefer the ones that run fast.
  - I don’t want an algorithm that takes a year to sort!
  - By the way, there are computations that run for months!
  - Operating systems on servers may run for years.
- We would like to compare algorithms based on their speed.
  - Mathematical model to capture algorithm efficiency.

Misconceptions

- Program P1 takes 10 seconds, P2 takes 20 seconds, so I would choose P1.
  - Execution time is input-dependent.
  - Execution time is hardware-dependent.
  - Execution time is machine-load dependent.
  - Execution time is run-dependent too!
  - Other factors play a role; for instance:

    whether the program is running in hostel or in DCF

    Or whether in Chennai or Kashmir

    Or whether in May or December!

Examples

```
a = a + b;
b = a – b;
a = a – b;
```

Irrespective of the values of a and b, this program would take time proportional to three instructions.

```
for (ii = 0; ii < N; ++ii)
    a[ii] = 0;
```

Proportional to N.

```
for (ii = 0; ii < N; ++ii)
    for (jj = 0; jj < M; ++jj)
        mat[ii][jj] = ii + jj;
```

Proportional to N*M.

```
int fun(int n) {
    return (n == 0 ? 1 : 4 * fun(n / 3));
}
```

?

Examples

```
a = a + b;
b = a – b;
a = a – b;
```

```
a[ii] = 0;
```

```
x = y;
if (x > 0)
    y = x + 1;
else
    z = x + 1;
```

```
for (ii = 0; ii < 1000; ++ii)
    a[ii] = 0;
```

All of these are equally efficient!

- They all perform constant-time operations.
- We denote those as O(1).

Examples

```
a[0] = 0;
a[1] = 0;
a[2] = 0;
...
a[n – 1] = 0;
```

```
int fact(int n) {
    return n * fact(n – 1);
}
```

```
for (ii = 0; ii < n; ++ii)
    a[ii] = 0;
```

All of these are equally efficient!

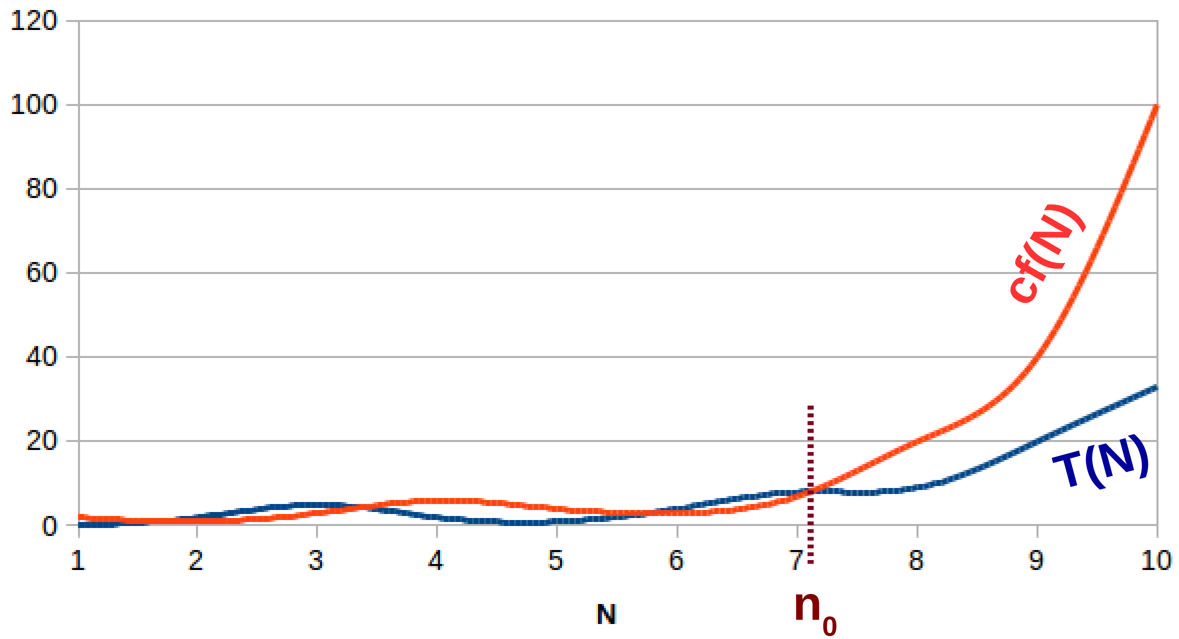
- They all perform linear-time operation *(linear in n)*.
- We denote those as O(n).

Definition

- $T(N) = O(1)$ if $T(N) \leq c$ when $N \geq n_0$, for some positive c and n_0.
- $T(N) = O(N)$ if $T(N) \leq cN$ when $N \geq n_0$, for some positive c and n_0.
- In general,

  $T(N) = O(f(N))$ if there exist positive constants c and n_0 such that $T(N) \leq cf(N)$ when $N \geq n_0$.
- Complexity captures the rate of growth of a function.

Big O



- In general,

  T(N) = O(f(N)) if there exist positive constants c and n_0 such that $T(N) \leq cf(N)$ when $N \geq n_0$.

- The complexity is upper-bounded by c*f(N).
- Thus, big O is the worst-case complexity.

Examples

```
a = a + b;
b = a – b;
a = a – b;
```

O(1)

Irrespective of the values of a and b, this program would take time proportional to three instructions.

```
for (ii = 0; ii < N; ++ii)
    a[ii] = 0;
```

O(N)

Proportional to N.

```
for (ii = 0; ii < N; ++ii)
    for (jj = 0; jj < M; ++jj)
        mat[ii][jj] = ii + jj;
```

O(N*M)

Proportional to N*M.

```
int fun(int n) {
    return (n == 0 ? 1 : 4 * fun(n / 3));
}
```

?

Big O as a Relation

- Recall from Discrete Mathematics
- O is reflexive: $T(n)$ is $O(T(n))$.
- O is transitive: If $T_1(n)$ is $O(T_2(n))$ and $T_2(n)$ is $O(T_3(n))$, then $T_1(n)$ is $O(T_3(n))$.
- O is not symmetric: $T_1(n)$ being $O(T_2(n))$ does <u>not</u> imply $T_2(n)$ is $O(T_1(n))$.

Types of Complexities

Symbol	Name	Bound	Equation
O(...)	Big O	Upper	$T(n) <= cf(n)$
Ω(...)	Big Omega	Lower	$T(n) >= cf(n)$
Θ(...)	Theta	Upper and Lower	$c_1f(n) <= T(n) <= c_2f(n)$
o(...)	Little O	Strictly Upper	$T(n) < cf(n)$
ω(...)	Little Omega	Strictly Lower	$T(n) > cf(n)$

Notes

- Θ means O and Ω. It is a stronger guarantee on the complexity.
- If T(n) is O(n), then T(n) is also O(n^2), also O(nlogn), also O(n^3), O(n^{100}), O(2^n); but it is not O(logn) or O(1).
- Big O is also called Big Oh.
- T(n) = T(n/2) = T(1000n) = T(nlog2) = T($2^{\log n}$)
- $\text{Log}_2(x)$, that is, *log to the base 2* is sometimes written as lg(x).
- If T(n) = O(f(n)) then f(n) = Ω(T(n)).

Theta as a Relation

- Recall from Discrete Mathematics
- Θ is reflexive: $T(n)$ is $\Theta(T(n))$.
- Θ is transitive: If $T_1(n)$ is $\Theta(T_2(n))$ and $T_2(n)$ is $\Theta(T_3(n))$, then $T_1(n)$ is $\Theta(T_3(n))$.
- Θ is symmetric: $T_1(n)$ being $\Theta(T_2(n))$ does imply $T_2(n)$ is $\Theta(T_1(n))$.
- Thus, complexity functions can be partitioned based on relation Θ.

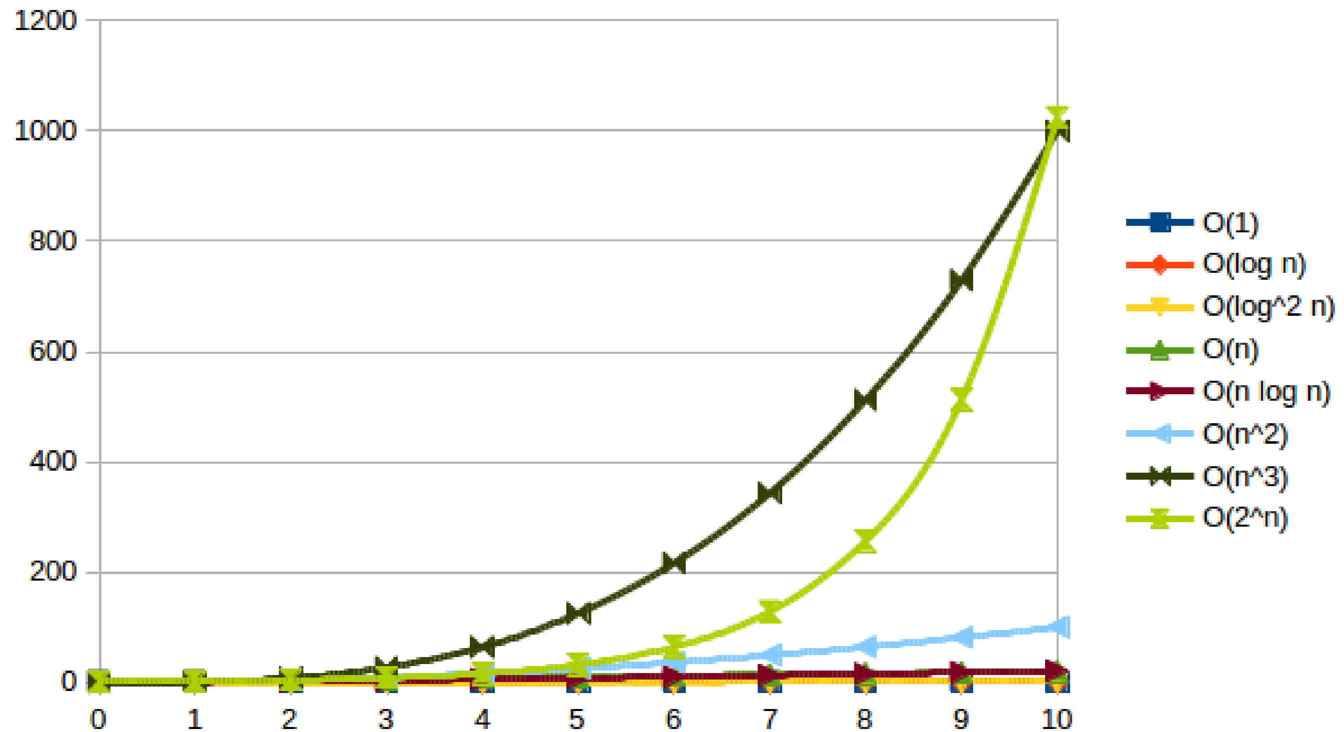
Complexity Arithmetic

- If T1(n) = O(f(n)) and T2(n) = O(g(n)), then
  - T1(n) + T2(n) = **max**(O(f(n), O(g(n)))
  - T1(n) * T2(n) = O(f(n) ***** g(n))

- **Classwork**:
  - Write a C code that requires the use of T1(n) + T2(n).
  - Write a C code that requires the use of T1(n) * T2(n).

Typical Complexities

Function	Name
c	Constant
Log N	Logarithmic
Log^2 N	Log-squared
N	Linear
N log N	Superlinear
N^2	Quadratic
N^3	Cubic
2^N	Exponential



Homework: Find which one grows faster: nlogn or $n^{1.5}$.

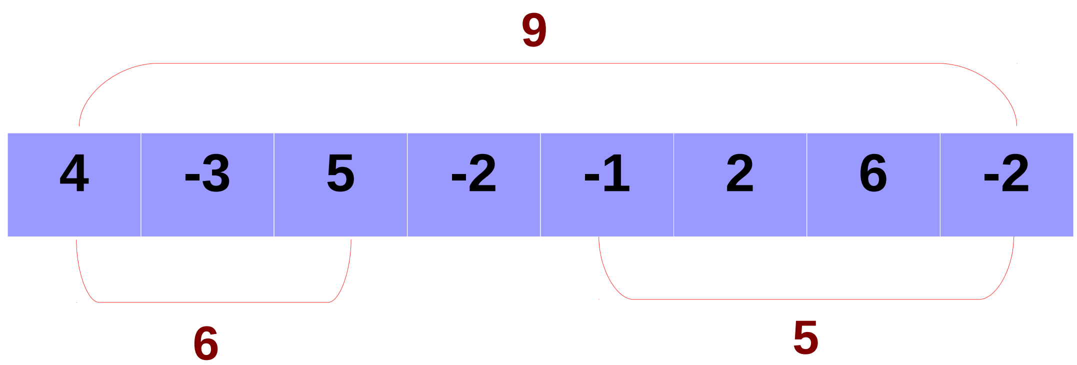
Complexity Comparison

- Given two complexity functions f(n) and g(n), we can determine relative growth rates using *$\lim_{n \to \infty} f(n) / g(n)$*, using L'Hospital's rule.
- Four possible values:
  - The limit is zero, implies f(n) = o(g(n)).
  - The limit is $c \neq 0$, implies f(n) = Θ(g(n)).
  - The limit is ∞, implies g(n) = o(f(n)).
  - The limit oscillates, implies there is no relation.

Facets of Efficiency

- An algorithm or its implementation may have various facets towards efficiency.
  - Time complexity (which we usually focus on)
  - Space complexity (considered in memory-critical systems such as embedded devices)
  - Energy complexity (e.g., your smartphones)
  - Security level (e.g., program with less versus more usage of pointers)
  - I/O complexity
  - ...

Max. Subsequence Sum



- **Problem Statement**

  Given an array of (positive, negative, zero) integer values, find the largest subsequence sum.

- A subsequence is a consecutive set of elements. If empty, its sum is zero.

MSS: Algorithm 1

Exhaustive Algorithm

For each possible subsequence

Compute sum

If sum > current maxsum

current maxsum = sum

Return current maxsum

How many subsequences?

What is the complexity of this part?

Algorithm 1 takes $O(N^3)$ running time.

Source: mss1.cpp

MSS: Algorithm 1

- Did we perform a tight mathematical analysis?
- To be precise, we need the following number of operations:

$$\sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \boxed{\sum_{k=i}^{j} O(1)}$$

$j - i + 1$

We will assume O(1) to be equal to constant 1. This would affect only the constant in BigOh.

MSS: Algorithm 1

- Did we perform a tight mathematical analysis?
- To be precise, we need the following number of operations:

$$\sum_{i=0}^{N-1} \underbrace{\boxed{\sum_{j=i}^{N-1} (j - i + 1)}}_{\substack{\text{sum of first N-i integers} \\ = \\ (N - i)(N - i + 1) / 2}}$$

MSS: Algorithm 1

- Did we perform a tight mathematical analysis?
- To be precise, we need the following number of operations:

$$\sum_{i=0}^{N-1} (N - i)(N - i + 1) / 2$$

$= (N^3 + 3N^2 + 2N) / 6$

$= O(N^3)$

The analysis is tight.
Is the algorithm tight?

MSS: Algorithm 2

- **Observation:**

$$\sum_{k=i}^{j} A[k] = A[j] + \sum_{k=i}^{j-1} A[k]$$

For each starting position i

For each ending position j

Incrementally compute sum

If sum > maxsum

maxsum = sum

Return maxsum

What is the complexity of this algorithm?

Source: mss2.cpp

MSS: Algorithm 3

- **Observation**: Discard fruitless subsequences early.

For each position

Add next element to sum

If sum > maxsum

Maxsum = sum

Else if sum is negative

sum = 0

Are you kidding?
This shouldn't work.
This is linear time algorithm!

It works due to the magic of **greedy algorithms**.

Source: mss3.cpp

Binary Search

- Go to page number 44.
- Searching in an array takes linear time O(N).
- If the array is sorted already, we can do better.
- We can cut the *search space* by half at every step.

5	**7**	**8**	**20**	**23**	**42**	**43**	**44**	**78**	**81**

Classwork: Write the code for binary search.
Source: **bsearch.cpp**

Binary Search

- Constant amount of time required to
  - Find the mid element.
  - Check if it is the element to be searched.
  - Decide whether to go to the left or the right.
  - Cut the search space by half.
- $T(N) = T(N/2) + O(1)$
  - Thus, $T(N)$ is $O(\log N)$.

Exercises

- Solve exercises at the end of Chapter 2 of Weiss’s book.