CS6015: Linear Algebra and Random Processes
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Exam - 1: Solutions

I. Multiple Choice Questions (Answer any eight)

1. Let V be a vector space with dimension 12. Let S be a subset of V which is linearly independent and has 11 vectors. Which of the following is FALSE?

    1. There must exist a linearly independent subset $S1$ of V such that $S \subsetneq S_1$ and S_1 is not a basis for V.
    2. Every nonempty subset S_1 of S is linearly independent.
    3. There must exist a linearly dependent subset S_1 of V such that $S \subsetneq S1$.
    4. Dimension of span(S) < dimension of V.

    Solution: (a)

2. Let W be a subspace of $\mathbb{R}^n$ and $W^\perp$ denote its orthogonal complement. If W_1 is subspace of $\mathbb{R}^n$ such that if $x \in W_1$, then $x^\intercal u = 0$, for all $u \in W^\perp$. Then,

    (a) dim $W_1^\perp \leq$ dim $W^\perp$
    (b) dim $W_1^\perp \leq$ dim W
    (c) dim $W_1^\perp \geq$ dim $W^\perp$
    (d) dim $W_1^\perp \geq$ dim W

    Solution: (c)

3. Let A be a 5×5 matrix with real entries and $x \neq 0$. Then, the vectors $x, Ax, A^2x, A^3x, A^4x, A^5x$ are

    1. linearly independent
    2. linearly dependent
    3. linearly independent if and only if A is symmetric
    4. linear dependence/independence cannot be determined from given data

    Solution: (b)

4. Let A, B be two complex $n \times n$ matrices that are Hermitian and

    $$C_1 = A + B, C_2 = iA + (2 + 3i)B, \text{ and } C_3 = AB.$$

    Then, among C_1, C_2, C_3, which is/are Hermitian?

    (a) Only C_1
    (b) Only C_2
    (c) Only C_3
    (d) All of them

    Solution: (a)

5. If A is a 10×8 real matrix with rank 8, then

    1. there exists at least one $b \in \mathbb{R}^{10}$ for which the system $Ax = b$ has infinite number of least square solutions.
    2. for every $b \in \mathbb{R}^{10}$, the system $Ax = b$ has infinite number of solutions.
    3. there exists at least one $b \in \mathbb{R}^{10}$ such that the system $Ax = b$ has a unique least square solution.
    4. for every $b \in \mathbb{R}^{10}$, the system $Ax = b$ has a unique solution.

Solution: (c)

6. Let A be a Hermitian matrix. Then, which of the following statements is false?
    1. The diagonal entries of A are all real.
    2. There exists a unitary U such that U^*AU is a diagonal matrix.
    3. If $A^3 = I$, then $A = I$.
    4. If $A^2 = I$, then $A = I$.

Solution: (d)

7. Let A be a complex $n \times n$ matrix. Let $\lambda_1, \lambda_2, \lambda_3$ be three distinct eigenvalues of A, with corresponding eigenvectors z_1, z_2, z_3. Then, which of the following statements is false?
    1. $z_1 + z_2$, $z_1 - z_2$, z_3 are linearly independent.
    2. z_1, z_2, z_3 are linearly independent.
    3. z_1, $z_1 + z_2$, $z_1 + z_2 + z_3$ are linearly independent.
    4. z_1, z_2, z_3 are linearly independent if and only if A is diagonalizable.

Solution: (d)

8. Let A be a $n \times n$ real matrix. Then, which of the following statements is true?
    1. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then A is similar to a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ along the diagonal.
    2. If $\operatorname{rank}(A) = r$, then A has r non-zero eigenvalues.
    3. If $A^k = 0$ for some $k > 0$, then $\operatorname{trace}(A) = 0$.
    4. If A has a repeated eigenvalue, then A is not diagonalizable.

Solution: (c)

9. Let P_1 and P_2 be $n \times n$ projection matrices. Then, which of the following statements is false?
    1. $P_1(P_1 - P_2)^2 = (P_1 - P_2)^2 P_1$ and $P_1(P_1 - P_2)^2 = (P_1 - P_2)^2 P_1$.
    2. Each eigenvalue of P_1 and P_2 is either 1 or 0.
    3. If P_1 and P_2 have the same rank, then they are similar.
    4. $\operatorname{rank}(P_1) + \operatorname{rank}(P_1 - I) \neq \operatorname{rank}(P_2) + \operatorname{rank}(P_2 - I)$.

Solution: (d)

II. True or False? (Answer any eight)

1. In $\mathbb{R}^9$, we can find a subspace W such that $\dim W = \dim W^\perp$.

Solution: False.

2. Let A and B be $n \times n$ real matrices. Then, $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

Solution: True.

3. If A is a $n \times n$ complex matrix with n orthonormal eigenvectors, then A is Hermitian.

Solution: False.

4. For any $a, b, c, d, e, f, g, h, i \in \mathbb{R}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}$ are similar.

Solution: True.

5. An $n \times n$ real matrix A is invertible if and only if the span of the rows of A is $\mathbb{R}^n$.

Solution: True.

6. The null space of A is equal to the null space of $A^\intercal A$.

Solution: True.

7. Let Q be a matrix with orthonormal columns. Then $QQ^\intercal = I$.

Solution: False.

8. Consider the vector space $\mathcal{M}$ of real 4×4 matrices. Then, the set of all invertible 4×4 matrices is a subspace of $\mathcal{M}$.

Solution: False.

9. Let A, B, C, D be square matrices of the same size. Then, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} D & C \\ B & A \end{vmatrix}$.

Solution: True.

10. If M and N are two subspaces of a vector space V and if every vector in V belongs either to M or to N (or both), then either $M = V$ or $N = V$ (or both).

Solution: True.

III. Problems that require detailed solutions **(Answer any four)**

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$. (3+2+3+2 marks)

   (a) Solve $Ax = 0$ and characterize the null space through its basis.

   (b) What is the rank of A? What are the dimensions of the column space, row space and left null space of A?

   (c) Find the complete solution of $Ax = b$, where $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

   (d) Find the conditions on b_1, b_2, b_3 that ensure $Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a solution.

Solution:

(a) Applying Gaussian elimination to the matrix A, we obtain

$$R = \begin{bmatrix} 1 & 2 & 0 & -2 & -3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two pivot variables and three free ones. Setting the free variables to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and calculating the pivot variable values gives us the following three vectors that form a basis for null space of A:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}. \qquad (1)$$

(b) Two pivot columns in R imply rank $(A) = 2$. The dimensions of the column space, row space and left null space of A are 2, 2 and 1, respectively.

(c) It is easy to see that $x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies $Ax_p = b$. For the complete solution add any linear combination of the vectors in (1) to x_p.

(d) Performing Gaussian elimination to the augmented matrix $[A : b]$, we obtain $[R : d]$, where $\begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \\ 2b_1 - b_2 + b_3 \end{bmatrix}$. The last row of R is zero, implying $2b_1 - b_2 + b_3 = 0$ to ensure $Ax = b$ has a solution.

2. Let W be a subspace of $\mathbb{R}^5$ defined as

$$W = \left\{ x \in \mathbb{R}^5 \mid x = \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \\ \alpha - \beta \\ \alpha + \beta \end{pmatrix}, \text{ where } \alpha, \beta \in \mathbb{R} \right\}.$$

Answer the following: (3+5+2 marks)

(a) Find a basis for W.

(b) Apply Gram-Schmidt procedure to the basis computed in the part above to get an orthonormal basis for W.

(c) Find the dimensions of W and $W^\perp$.

Solution:

(a) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$

(b) Starting with first basis vector, Gram-Schmidt would give $\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{60}}\begin{bmatrix} -1 \\ 4 \\ 3 \\ -5 \\ 3 \end{bmatrix}.$

Starting with the second basis vector, the orthonormal basis would come out as $\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{60}}\begin{bmatrix} 4 \\ -1 \\ 3 \\ 5 \\ 3 \end{bmatrix}.$

(c) From part (a), dimension of W is 2, implying dimension of $W^\perp$ is 3.

3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Answer the following: (8+2 marks)

(a) Given that A has an eigenvalue 1 with corresponding eigenvector $x_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find the Schur decomposition of A, i.e., find a matrix P with orthonormal columns such that $P^\intercal AP$ is upper-triangular.

(b) Is A diagonalizable? Justify your answer.

[(a)]

Solution:

(a) Any vector of the form $\begin{bmatrix} \alpha \\ -2\alpha \\ \beta \end{bmatrix}, \alpha, \beta \in \mathbb{R}$ is orthogonal to x_1.

Letting $x_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we obtain an orthonormal basis $\{x_1, x_2, x_3\}$.
Set P to be a matrix with x_1, x_2 and x_3 as its columns. Then,

$$P^\intercal AP = \begin{bmatrix} 1 & 1 & \frac{2}{\sqrt{5}} \\ 0 & -1 & \frac{1}{\sqrt{5}} \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) A is a real matrix, so $A^* = A^\intercal$. Notice that $AA^\intercal \neq A^\intercal A$. Thus, A is not a normal matrix and hence, not diagonalizable.

4. Let $A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$. Answer the following: (3+5+2 marks)

(a) Find all eigenvectors of A. Is A diagonalizable, i.e., does there exist an invertible S such that $S^{-1}AS$ is diagonal? Justify your answer.

(b) Compute the SVD of A, i.e., find Q_1, Σ, Q_2 such that $A = Q_1 \Sigma Q_2^\intercal$, where Q_1, Q_2 orthogonal and Σ is a diagonal matrix with non-negative entries along the diagonal.

(c) Find a matrix B that is similar to A, but not the same as A.

Solution:

(a) A has eigenvalue $\sqrt{2}$ repeated twice. Since $A - \sqrt{2}I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for A and there aren't any more independent ones. Hence, A is not diagonalizable.

(b) $A^\intercal A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$ has characteristic polynomial $(\lambda - 4)(\lambda - 1)$. Thus, the singular values are $\sigma_1 = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{1} = 1$ and hence $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Next, we find the eigenvectors of $A^\intercal A$. Observe that $A^\intercal A - 4I = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$ and hence, $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$ is an eigenvector. Normalizing, we get $u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$. Along similar lines, $u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$ is another independent eigenvector for the null space of $A^\intercal A - I$. These eigenvectors go into the Q_2 matrix, i.e., $Q_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$.

Suppose the vectors v_1 and v_2 are the columns of the matrix Q_1. Then, $\sigma_1 v_1 = Au_1$ and $\sigma_1 v_2 = Au_2$, leading to $v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$ and $v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$. Hence,

$$A = \underbrace{\left(\frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{bmatrix} \right)}_{Q_1} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}^\intercal \right)}_{Q_2^\intercal}.$$

(c) Take any invertible matrix S and set $B = SAS^{-1}$.

5. The following information about a 5×4 real matrix A is available:

- The characteristic polynomial of $A^\intercal A$ is $(\lambda - 9)(\lambda - 4)\lambda^2$.
- $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ are the eigenvectors corresponding to $\lambda_1 = 9$ and $\lambda_2 = 4$ of $A^\intercal A$.
- $Aq_1 = \sqrt{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $Aq_2 = \sqrt{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Using the above information, (8 + 2 marks)

(a) find the matrix A.

(b) find a basis for null space of A.

Solution:

(a) From the characteristic polynomial of $A^\intercal A$, the singular values can be read off as $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$. The full-rank SVD of A would be of the form

$$A = \sigma_1 v_1 q_1^\intercal + \sigma_2 v_2 q_2^\intercal.$$

To find v_1, v_2, observe that $\sigma_1 v_1 = Aq_1$ and $\sigma_2 v_2 = Aq_2$, leading to

$$\begin{aligned} A &= Aq_1 q_1^\intercal + Aq_2 q_2^\intercal \\ &= \sqrt{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(b) Extend q_1, q_2 to an orthonormal basis of $\mathbb{R}^4$. For this, observe that the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set of vectors.

Normalizing, we obtain, $\{q_1, q_2, q_3, q_4\}$, where $q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$ and $q_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Then, the set $\{q_3, q_4\}$ would be a basis for the null space of A (Why?).