CS6015: Linear Algebra and Random Processes
Exam - 1
Course Instructor : Prashanth L.A.
Date : Sep-24, 2017 **Duration** : 150 minutes

Name of the student :
Roll No :

INSTRUCTIONS: The test will be evaluated ONLY based on what is written on the question paper. Write all your FINAL answers in the question paper and submit it at the end of the test. The test is divided into 3 sections - Multiple Choice Questions, True or False type questions and Questions for Detailed Working. For the first three sections, please provide the final answer ONLY. For the last section, final answers along with proper justification need to be provided. Please use rough sheets for any calculations. Please DO NOT submit the rough sheets.

I. Multiple Choice Questions (Answer any eight)

Note: $1\frac{1}{2}$ marks for the correct answer and $-\frac{1}{2}$ for the wrong answer. Only one answer is correct. Please write the choice code a, b, c or d in the answer box provided.

1. Let V be a vector space with dimension 12. Let S be a subset of V which is linearly independent and has 11 vectors. Which of the following is FALSE?

   (a) There must exist a linearly independent subset S_1 of V such that $S \subsetneq S_1$ and S_1 is not a basis for V.

   (b) Every nonempty subset S_1 of S is linearly independent.

   (c) There must exist a linearly dependent subset S_1 of V such that $S \subsetneq S_1$.

   (d) Dimension of span(S) < dimension of V.

   Answer: $\boxed{}$

2. Let W be a subspace of $\mathbb{R}^n$ and $W^\perp$ denote its orthogonal complement. If W_1 is subspace of $\mathbb{R}^n$ such that if $x \in W_1$, then $x^\intercal u = 0$, for all $u \in W^\perp$. Then,

   (a) dim $W_1^\perp \leq$ dim $W^\perp$ (b) dim $W_1^\perp \leq$ dim W

   (c) dim $W_1^\perp \geq$ dim $W^\perp$ (d) dim $W_1^\perp \geq$ dim W

   Answer: $\boxed{}$

3. Let A be a 5×5 matrix with real entries and $x \neq 0$. Then, the vectors $x, Ax, A^2x, A^3x, A^4x, A^5x$ are

   (a) linearly independent

   (b) linearly dependent

   (c) linearly independent if and only if A is symmetric

   (d) linear dependence/independence cannot be determined from given data

   Answer: $\boxed{}$

4. Let A, B be two complex $n \times n$ matrices that are Hermitian and

   $$C_1 = A + B, C_2 = iA + (2 + 3i)B, \text{ and } C_3 = AB.$$

   Then, among C_1, C_2, C_3, which is/are Hermitian?

(a) Only C_1 (b) Only C_2

(c) Only C_3 (d) All of them

Answer: $\square$

5. If A is a 10×8 real matrix with rank 8, then

   (a) there exists at least one $b \in \mathbb{R}^{10}$ for which the system $Ax = b$ has infinite number of least square solutions.

   (b) for every $b \in \mathbb{R}^{10}$, the system $Ax = b$ has infinite number of solutions.

   (c) there exists at least one $b \in \mathbb{R}^{10}$ such that the system $Ax = b$ has a unique least square solution.

   (d) for every $b \in \mathbb{R}^{10}$, the system $Ax = b$ has a unique solution.

   Answer: $\square$

6. Let A be a Hermitian matrix. Then, which of the following statements is false?

   (a) The diagonal entries of A are all real.

   (b) There exists a unitary U such that U^*AU is a diagonal matrix.

   (c) If $A^3 = I$, then $A = I$.

   (d) If $A^2 = I$, then $A = I$.

   Answer: $\square$

7. Let A be a complex $n \times n$ matrix. Let $\lambda_1, \lambda_2, \lambda_3$ be three distinct eigenvalues of A, with corresponding eigenvectors z_1, z_2, z_3. Then, which of the following statements is false?

   (a) $z_1 + z_2$, $z_1 - z_2$, z_3 are linearly independent.

   (b) z_1, z_2, z_3 are linearly independent.

   (c) z_1, $z_1 + z_2$, $z_1 + z_2 + z_3$ are linearly independent.

   (d) z_1, z_2, z_3 are linearly independent if and only if A is diagonalizable.

   Answer: $\square$

8. Let A be a $n \times n$ real matrix. Then, which of the following statements is true?

   (a) If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then A is similar to a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ along the diagonal.

   (b) If $\operatorname{rank}(A) = r$, then A has r non-zero eigenvalues.

   (c) If $A^k = 0$ for some $k > 0$, then $\operatorname{trace}(A) = 0$.

   (d) If A has a repeated eigenvalue, then A is not diagonalizable.

   Answer: $\square$

9. Let P_1 and P_2 be $n \times n$ projection matrices. Then, which of the following statements is false?

   (a) $P_1(P_1 - P_2)^2 = (P_1 - P_2)^2 P_1$ and $P_1(P_1 - P_2)^2 = (P_1 - P_2)^2 P_1$.

   (b) Each eigenvalue of P_1 and P_2 is either 1 or 0.

   (c) If P_1 and P_2 have the same rank, then they are similar.

   (d) $\operatorname{rank}(P_1) + \operatorname{rank}(P_1 - I) \neq \operatorname{rank}(P_2) + \operatorname{rank}(P_2 - I)$.

   Answer: $\square$

II. True or False? **(Answer any eight)**

Note: 1 *marks for the correct answer and* $-\frac{1}{2}$ *for the wrong answer.*

1. In $\mathbb{R}^9$, we can find a subspace W such that $\dim W = \dim W^{\perp}$.

   Answer: $\square$

2. Let A and B be $n \times n$ real matrices. Then, $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

   Answer: $\square$

3. If A is a $n \times n$ complex matrix with n orthonormal eigenvectors, then A is Hermitian.

   Answer: $\square$

4. For any $a, b, c, d, e, f, g, h, i \in \mathbb{R}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}$ are similar.

   Answer: $\square$

5. An $n \times n$ real matrix A is invertible if and only if the span of the rows of A is $\mathbb{R}^n$.

   Answer: $\square$

6. The null space of A is equal to the null space of $A^{\intercal}A$.

   Answer: $\square$

7. Let Q be a matrix with orthonormal columns. Then $QQ^{\intercal} = I$.

   Answer: $\square$

8. Consider the vector space $\mathcal{M}$ of real 4×4 matrices. Then, the set of all invertible 4×4 matrices is a subspace of $\mathcal{M}$.

   Answer: $\square$

9. Let A, B, C, D be square matrices of the same size. Then, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} D & C \\ B & A \end{vmatrix}$.

   Answer: $\square$

10. If M and N are two subspaces of a vector space V and if every vector in V belongs either to M or to N (or both), then either $M = V$ or $N = V$ (or both).

    Answer: $\square$

III. Problems that require detailed solutions (Answer any four)

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$. (3+2+3+2 marks)

   (a) Solve $Ax = 0$ and characterize the null space through its basis.

   (b) What is the rank of A? What are the dimensions of the column space, row space and left null space of A?

   (c) Find the complete solution of $Ax = b$, where $b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

   (d) Find the conditions on b_1, b_2, b_3 that ensure $Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a solution.

2. Let W be a subspace of $\mathbb{R}^5$ defined as

$$W = \left\{ x \in \mathbb{R}^5 \mid x = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \\ \alpha-\beta \\ \alpha+\beta \end{pmatrix}, \text{ where } \alpha, \beta \in \mathbb{R} \right\}.$$

Answer the following: (3+5+2 marks)

(a) Find a basis for W.

(b) Apply Gram-Schmidt procedure to the basis computed in the part above to get an orthonormal basis for W.

(c) Find the dimensions of W and $W^{\perp}$.

3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Answer the following: (8+2 marks)

(a) Given that A has an eigenvalue 1 with corresponding eigenvector $x_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find the Schur decomposition of A, i.e., find a matrix P with orthonormal columns such that $P^\top AP$ is upper-triangular.

(b) Is A diagonalizable? Justify your answer.

4. Let $A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$. Answer the following: (3+5+2 marks)

(a) Find all eigenvectors of A. Is A diagonalizable, i.e., does there exist an invertible S such that $S^{-1}AS$ is diagonal? Justify your answer.

(b) Compute the SVD of A, i.e., find Q_1, Σ, Q_2 such that $A = Q_1 \Sigma Q_2^\intercal$, where Q_1, Q_2 orthogonal and Σ is a diagonal matrix with non-negative entries along the diagonal.

(c) Find a matrix B that is similar to A, but not the same as A.

5. The following information about a 5×4 real matrix A is available:

- The characteristic polynomial of $A^{\intercal}A$ is $(\lambda - 9)(\lambda - 4)\lambda^2$.
- $q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ are the eigenvectors corresponding to $\lambda_1 = 9$ and $\lambda_2 = 4$ of $A^{\intercal}A$.
- $Aq_1 = \sqrt{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $Aq_2 = \sqrt{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Using the above information, (8 + 2 marks)

(a) find the matrix A.

(b) find a basis for null space of A.