CS7015 (Deep Learning) : Lecture 14

Sequence Learning Problems, Recurrent Neural Networks, Backpropagation Through Time (BPTT), Vanishing and Exploding Gradients, Truncated BPTT

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Module 14.1: Sequence Learning Problems

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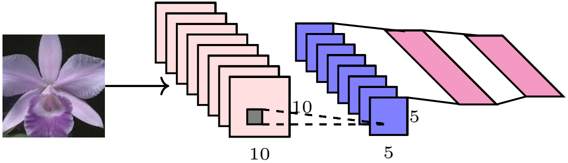

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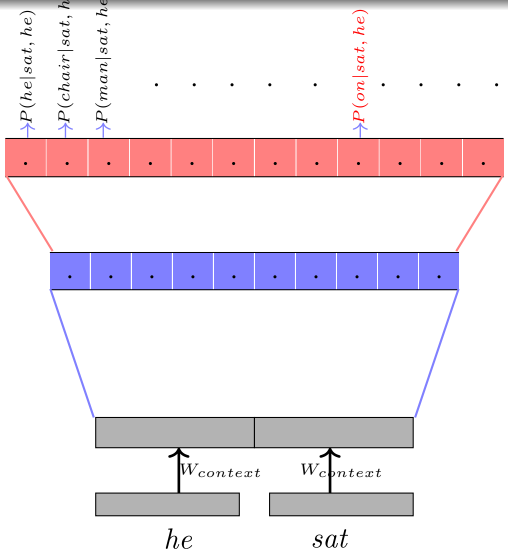

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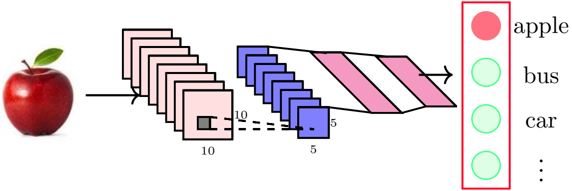

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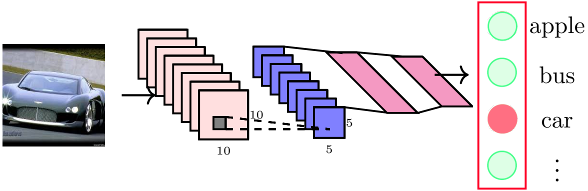

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- For example, the computatations, outputs and decisions for two successive images are completely independent of each other

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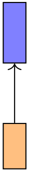

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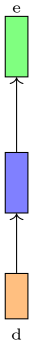
- Given the first character 'd' you want to predict the next character 'e' and so on



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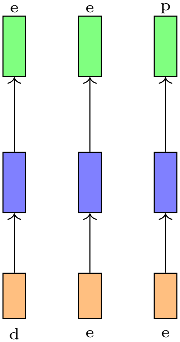

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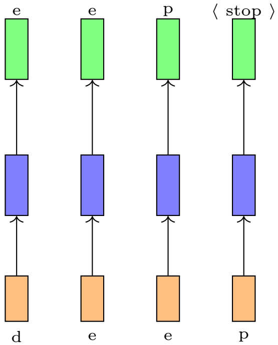

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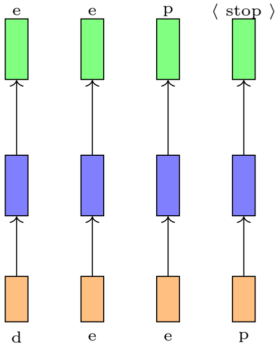

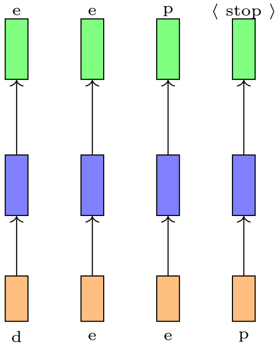

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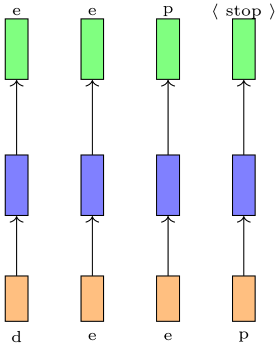
- First, successive inputs are no longer independent (while predicting 'e' you would want to know what the previous input was in addition to the current input)



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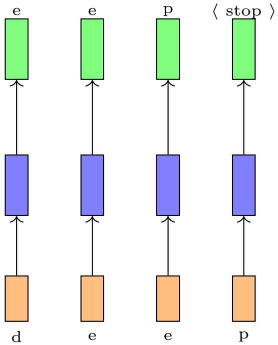

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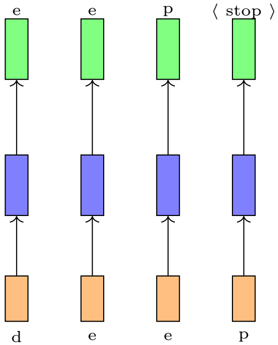
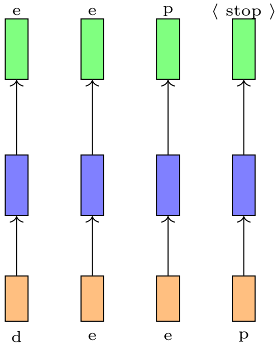
- First, successive inputs are no longer independent (while predicting 'e' you would want to know what the previous input was in addition to the current input)
- Second, the length of the inputs and the number of predictions you need to make is not fixed (for example, "learn", "deep", "machine" have different number of characters)
- Third, each network (orange-blue-green structure) is performing the same task (**input** : character **output** : character)

- These are known as sequence learning problems





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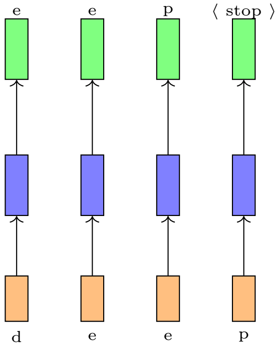

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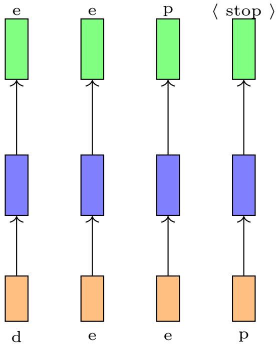

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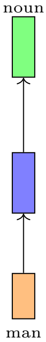
- Let us look at some more examples of such problems

- Consider the task of predicting the part of speech tag (noun, adverb, adjective verb) of each word in a sentence



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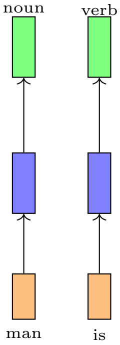

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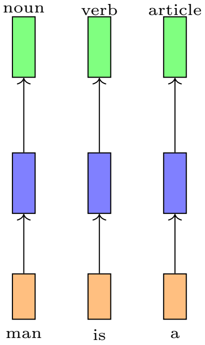

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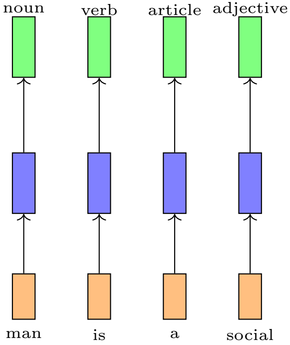

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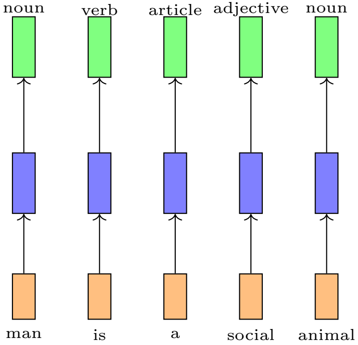

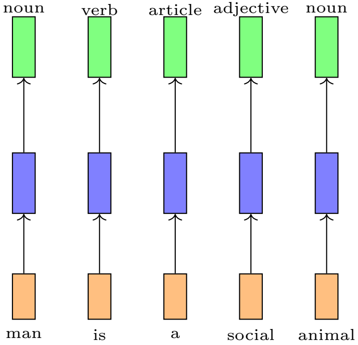

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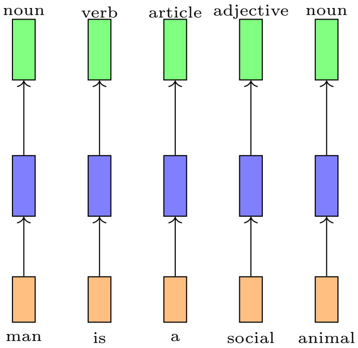

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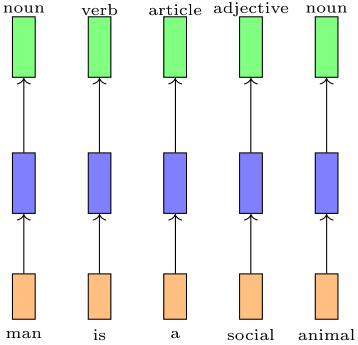

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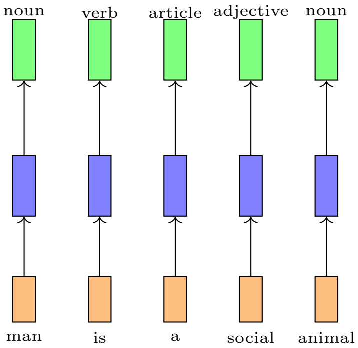

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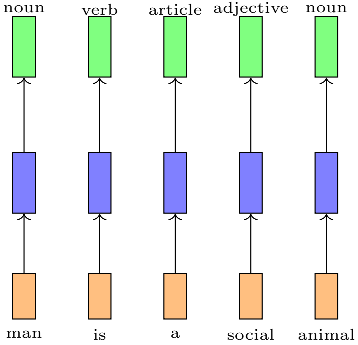

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- Each network is performing the same task (**input** : word, **output** : tag)

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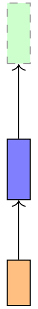

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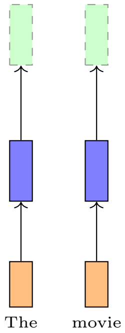

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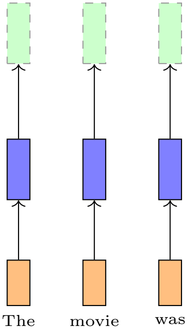

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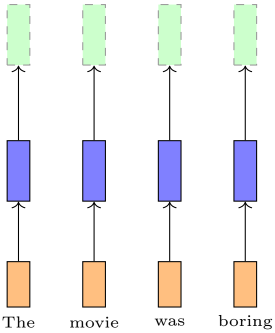

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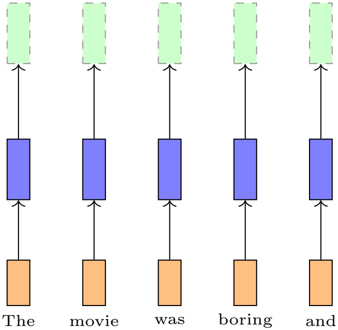

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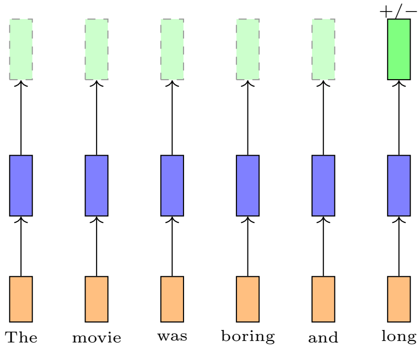

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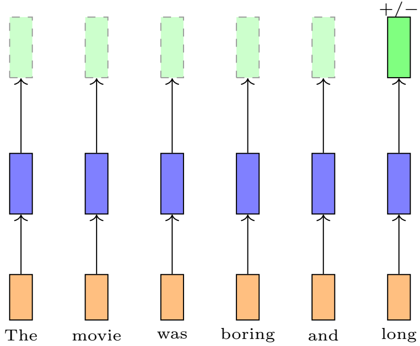

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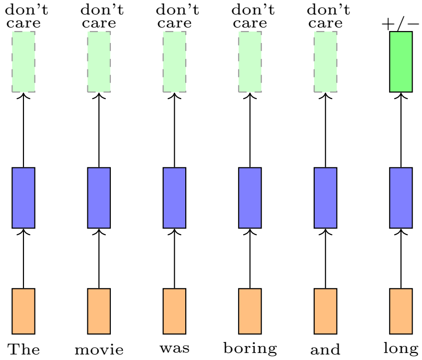

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- Here again we could think that the network is performing the same task at each step (input : word, output : +/−) but it's just that we don't care about intermediate outputs

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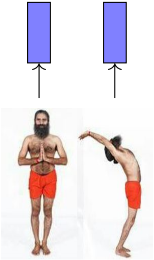

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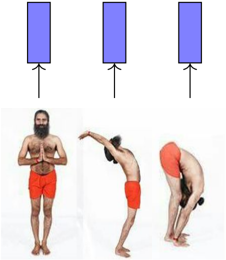

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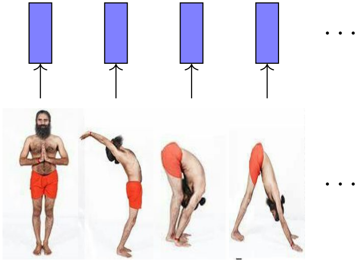

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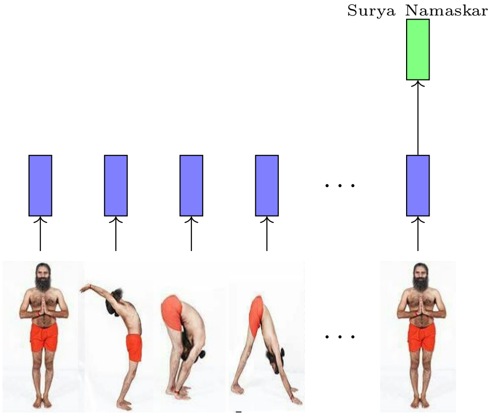

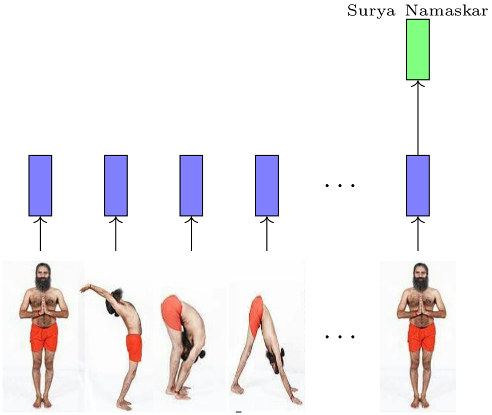

- Sequences could be composed of anything (not just words)
- For example, a video could be treated as a sequence of images
- We may want to look at the entire sequence and detect the activity being performed

Module 14.2: Recurrent Neural Networks

How do we model such tasks involving sequences ?

Wishlist

- Account for dependence between inputs

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- Account for dependence between inputs
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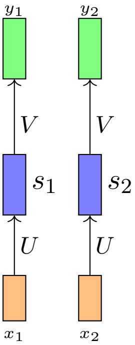
- We will focus on each of these to arrive at a model for dealing with sequences

- What is the function being executed at each time step ?



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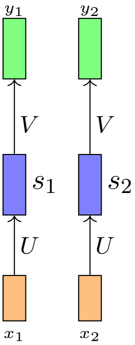
$$s_i = \sigma(Ux_i + b)$$



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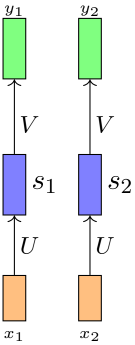

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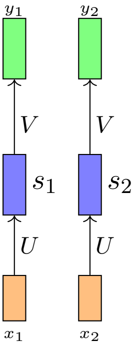

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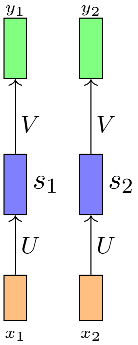
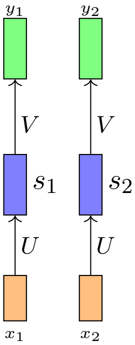
- Since we want the same function to be executed at each timestep we should share the same network (i.e., same parameters at each timestep)



- This parameter sharing also ensures that the network becomes agnostic to the length (size) of the input



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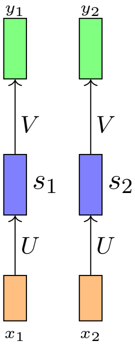

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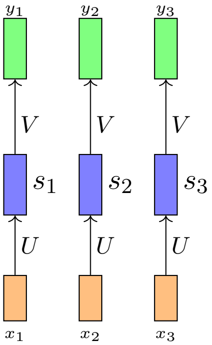
- We just create multiple copies of the network and execute them at each timestep



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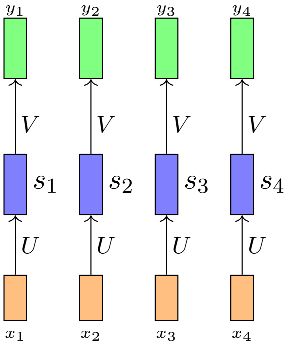

- This parameter sharing also ensures that the network becomes agnostic to the length (size) of the input
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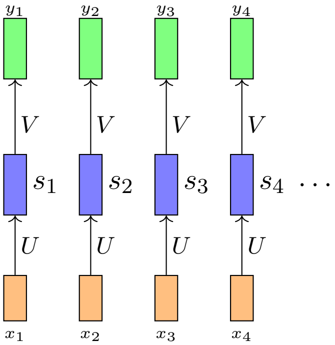

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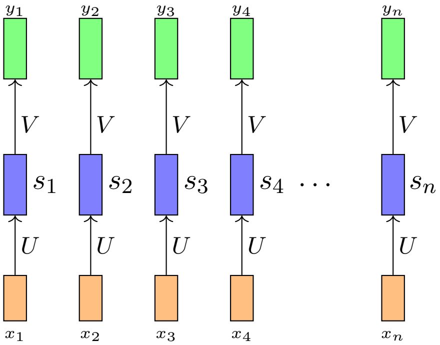

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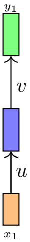
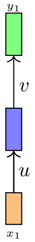
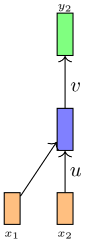
- How do we account for dependence between inputs ?

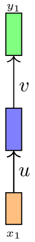


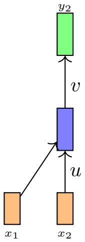
- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this

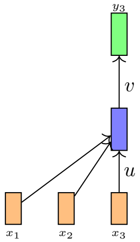




- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this
- At each timestep we will feed all the previous inputs to the network







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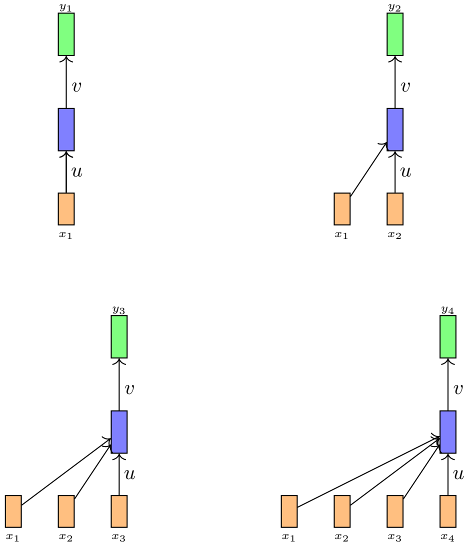

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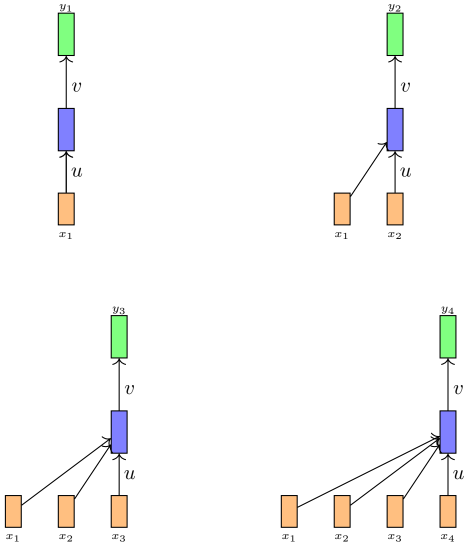
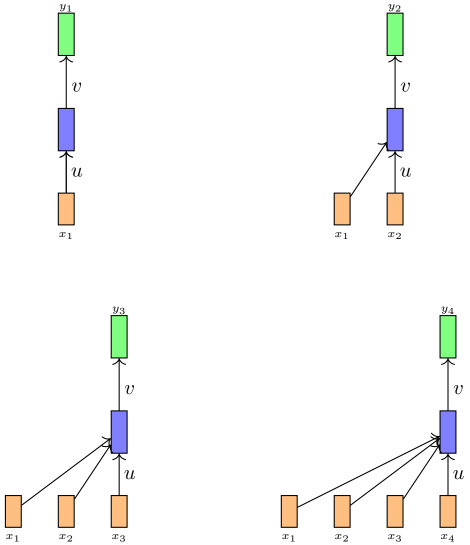
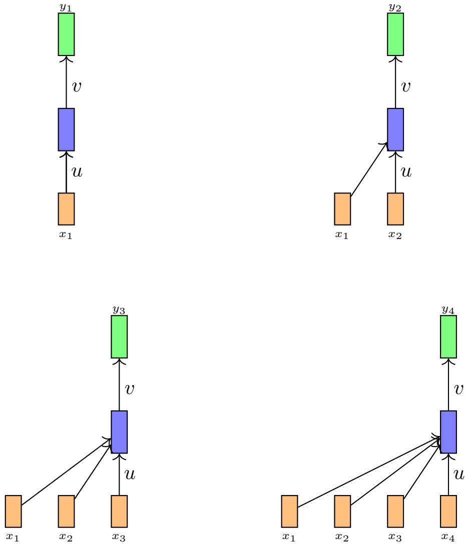


- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this
- At each timestep we will feed all the previous inputs to the network
- Is this okay ?



- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this
- At each timestep we will feed all the previous inputs to the network
- Is this okay ?
- No, it violates the other two items on our wishlist



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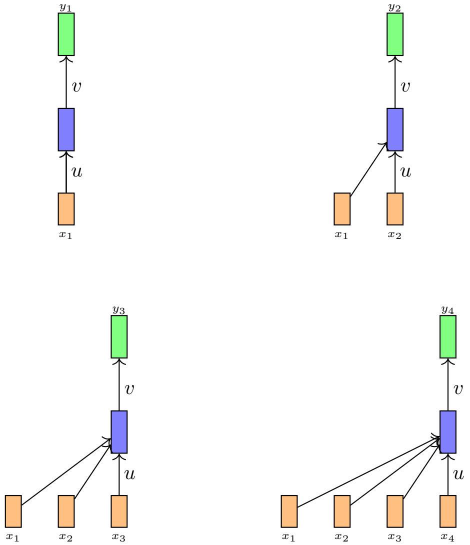
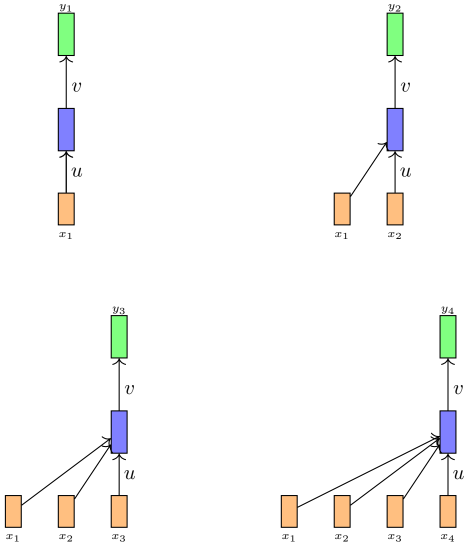
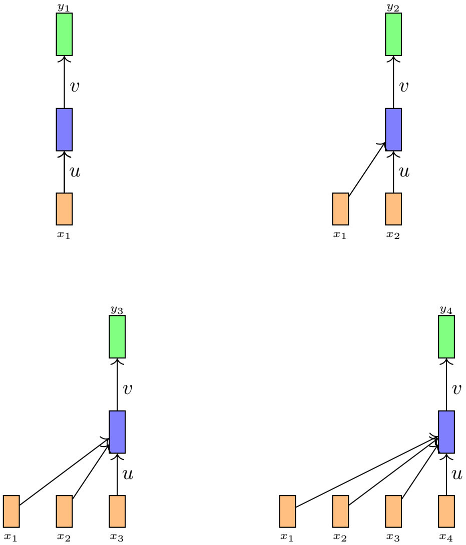
- No, it violates the other two items on our wishlist
- How ?



- How do we account for dependence between inputs ?
- Let us first see an infeasible way of doing this
- At each timestep we will feed all the previous inputs to the network
- Is this okay ?
- No, it violates the other two items on our wishlist
- How ? Let us see

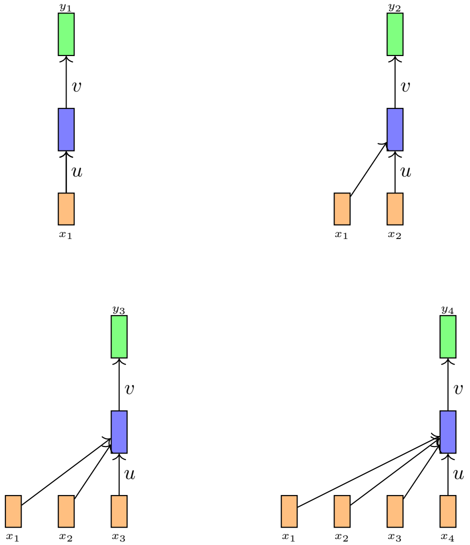


- First, the function being computed at each time-step now is different



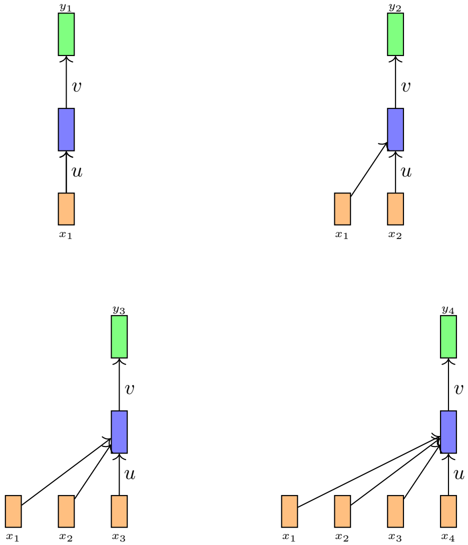
- First, the function being computed at each time-step now is different

$$y_1 = f_1(x_1)$$



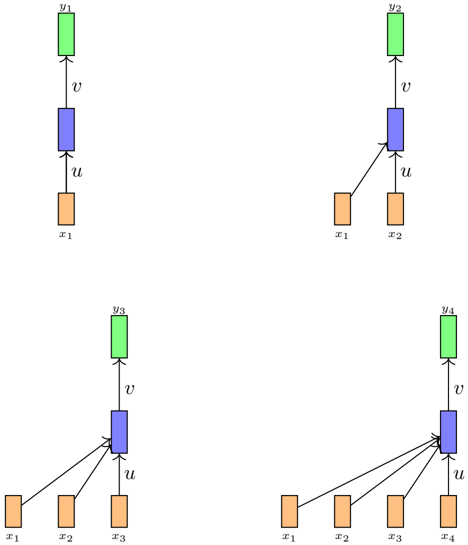
- First, the function being computed at each time-step now is different

$$y_1 = f_1(x_1)$$
$$y_2 = f_2(x_1, x_2)$$



- First, the function being computed at each time-step now is different

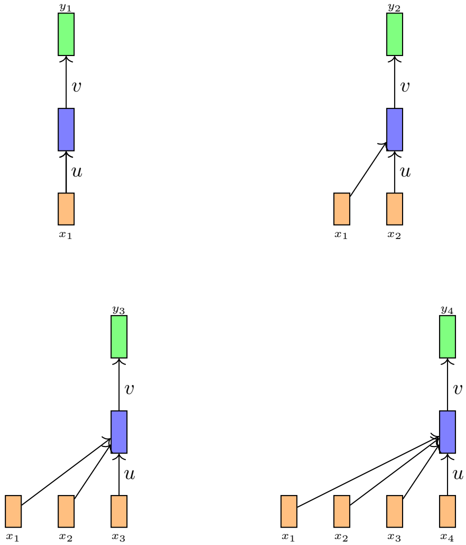
$$
\begin{aligned}
y_1 &= f_1(x_1) \\
y_2 &= f_2(x_1, x_2) \\
y_3 &= f_3(x_1, x_2, x_3)
\end{aligned}
$$



- First, the function being computed at each time-step now is different

$$y_1 = f_1(x_1)$$
$$y_2 = f_2(x_1, x_2)$$
$$y_3 = f_3(x_1, x_2, x_3)$$

- The network is now sensitive to the length of the sequence

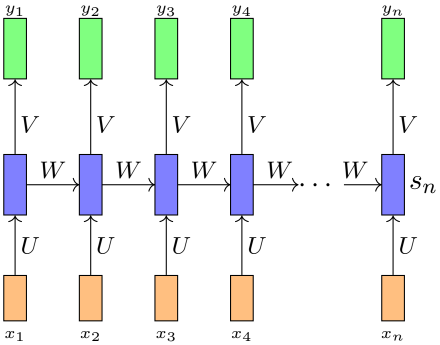


- First, the function being computed at each time-step now is different

$$y_1 = f_1(x_1)$$
$$y_2 = f_2(x_1, x_2)$$
$$y_3 = f_3(x_1, x_2, x_3)$$

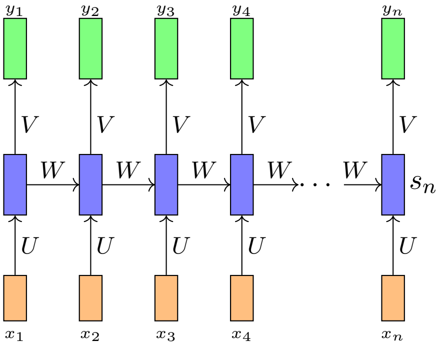
- The network is now sensitive to the length of the sequence
- For example a sequence of length 10 will require $f_1, \ldots, f_{10}$ whereas a sequence of length 100 will require $f_1, \ldots, f_{100}$

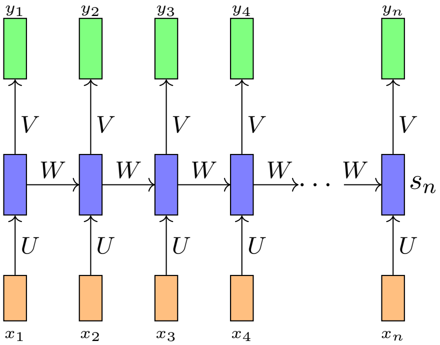
- The solution is to add a recurrent connection in the network,



- The solution is to add a recurrent connection in the network,

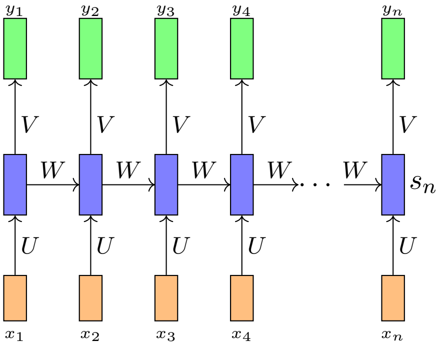
$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$





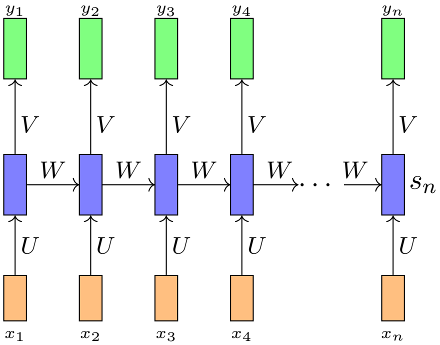
- The solution is to add a recurrent connection in the network,

$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
$$y_i = \mathcal{O}(Vs_i + c)$$



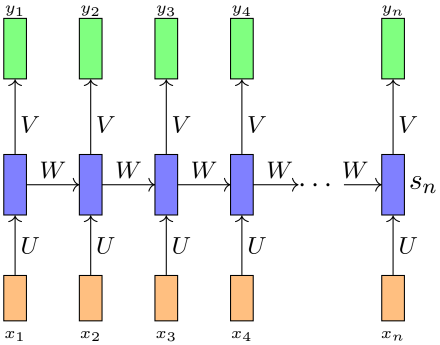
- The solution is to add a recurrent connection in the network,

$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
$$y_i = \mathcal{O}(Vs_i + c)$$
$$or$$



- The solution is to add a recurrent connection in the network,

$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
$$y_i = \mathcal{O}(Vs_i + c)$$
$$or$$
$$y_i = f(x_i, s_{i-1}, W, U, V, b, c)$$



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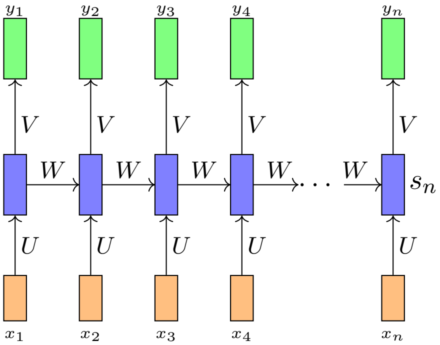
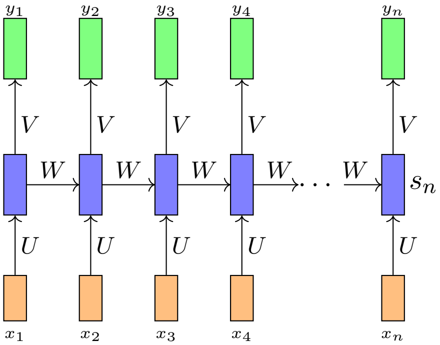
- s_i is the state of the network at timestep i



- The solution is to add a recurrent connection in the network,

$$s_i = \sigma(Ux_i + Ws_{i-1} + b)$$
$$y_i = \mathcal{O}(Vs_i + c)$$
$$or$$
$$y_i = f(x_i, s_{i-1}, W, U, V, b, c)$$

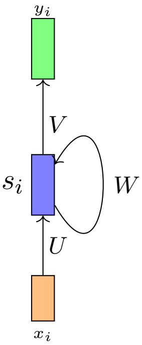
- s_i is the state of the network at timestep i
- The parameters are W, U, V, c, b which are shared across timesteps



- The solution is to add a recurrent connection in the network,

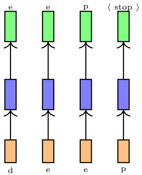
$$
\begin{aligned}
s_i &= \sigma(Ux_i + Ws_{i-1} + b) \\
y_i &= \mathcal{O}(Vs_i + c) \\
&\quad or \\
y_i &= f(x_i, s_{i-1}, W, U, V, b, c)
\end{aligned}
$$

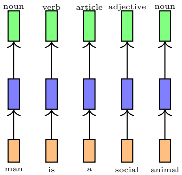
- s_i is the state of the network at timestep i
- The parameters are W, U, V, c, b which are shared across timesteps
- The same network (and parameters) can be used to compute $y_1, y_2, \ldots, y_{10}$ or y_{100}

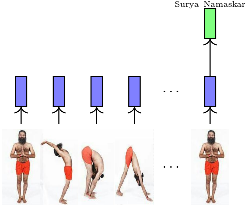


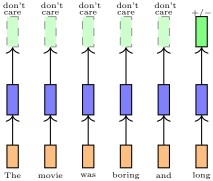
- This can be represented more compactly

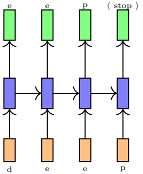
- Let us revisit the sequence learning problems that we saw earlier

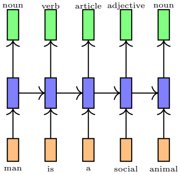


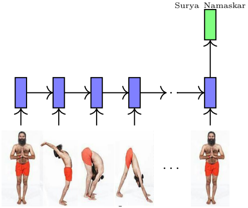


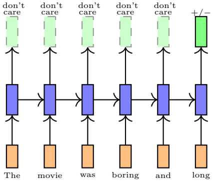






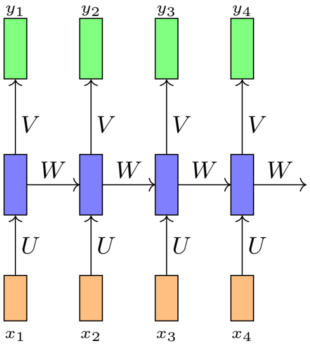




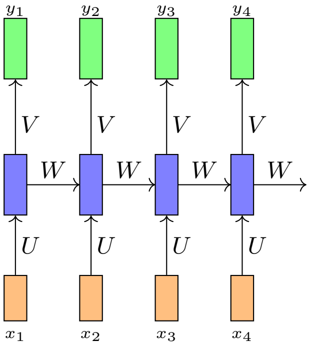


- Let us revisit the sequence learning problems that we saw earlier
- We now have recurrent connections between time steps which account for dependence between inputs

Module 14.3: Backpropagation through time

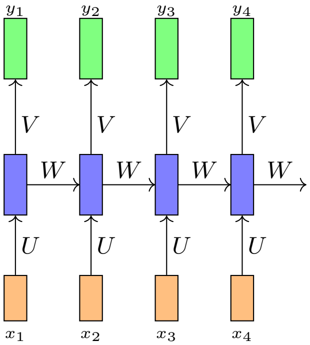


- Before proceeding let us look at the dimensions of the parameters carefully



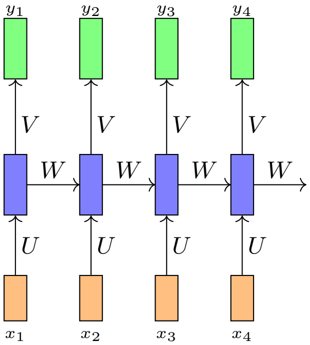
- Before proceeding let us look at the dimensions of the parameters carefully

$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$



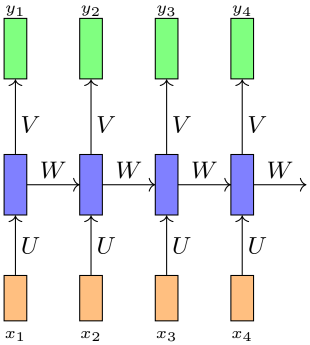
- Before proceeding let us look at the dimensions of the parameters carefully

$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$
$$s_i \in \mathbb{R}^d \quad \text{(d-dimensional state)}$$



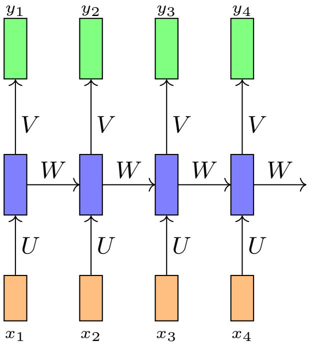
- Before proceeding let us look at the dimensions of the parameters carefully

$$
\begin{aligned}
x_i &\in \mathbb{R}^n && \text{(n-dimensional input)} \\
s_i &\in \mathbb{R}^d && \text{(d-dimensional state)} \\
y_i &\in \mathbb{R}^k && \text{(say k classes)}
\end{aligned}
$$



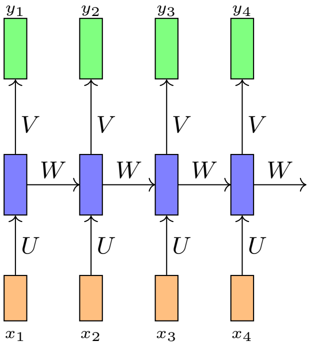
- Before proceeding let us look at the dimensions of the parameters carefully

$$
\begin{aligned}
x_i &\in \mathbb{R}^n \quad \text{(n-dimensional input)} \\
s_i &\in \mathbb{R}^d \quad \text{(d-dimensional state)} \\
y_i &\in \mathbb{R}^k \quad \text{(say k classes)} \\
U &\in
\end{aligned}
$$



- Before proceeding let us look at the dimensions of the parameters carefully

$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$
$$s_i \in \mathbb{R}^d \quad \text{(d-dimensional state)}$$
$$y_i \in \mathbb{R}^k \quad \text{(say k classes)}$$
$$U \in \mathbb{R}^{n \times d}$$



- Before proceeding let us look at the dimensions of the parameters carefully

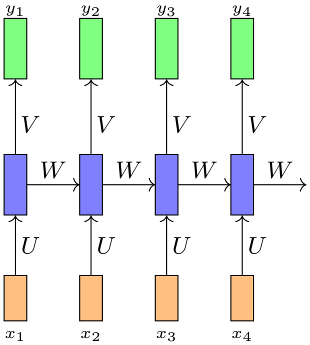
$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$

$$s_i \in \mathbb{R}^d \quad \text{(d-dimensional state)}$$

$$y_i \in \mathbb{R}^k \quad \text{(say k classes)}$$

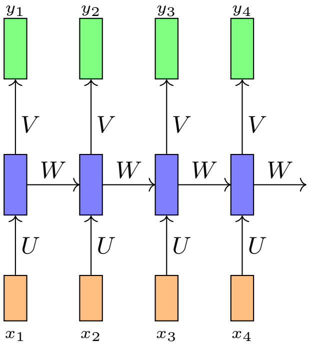
$$U \in \mathbb{R}^{n \times d}$$

$$V \in$$



- Before proceeding let us look at the dimensions of the parameters carefully

$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$
$$s_i \in \mathbb{R}^d \quad \text{(d-dimensional state)}$$
$$y_i \in \mathbb{R}^k \quad \text{(say k classes)}$$
$$U \in \mathbb{R}^{n \times d}$$
$$V \in \mathbb{R}^{d \times k}$$



- Before proceeding let us look at the dimensions of the parameters carefully

$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$

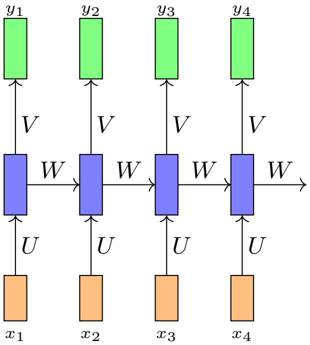
$$s_i \in \mathbb{R}^d \quad \text{(d-dimensional state)}$$

$$y_i \in \mathbb{R}^k \quad \text{(say k classes)}$$

$$U \in \mathbb{R}^{n \times d}$$

$$V \in \mathbb{R}^{d \times k}$$

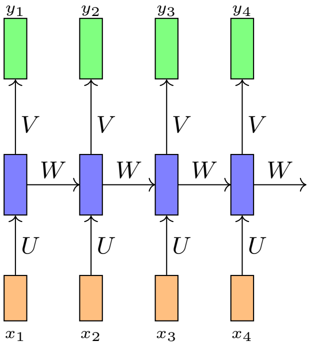
$$W \in$$

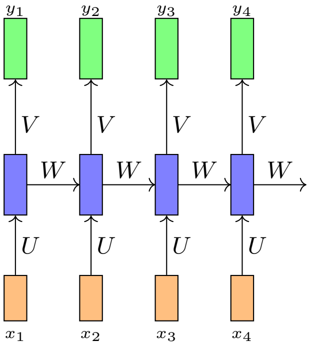


- Before proceeding let us look at the dimensions of the parameters carefully

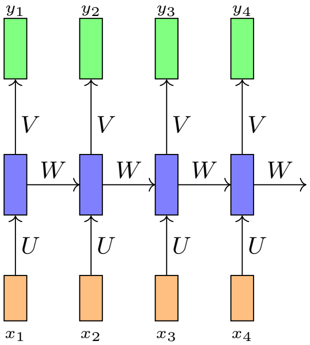
$$x_i \in \mathbb{R}^n \quad \text{(n-dimensional input)}$$
$$s_i \in \mathbb{R}^d \quad \text{(d-dimensional state)}$$
$$y_i \in \mathbb{R}^k \quad \text{(say k classes)}$$
$$U \in \mathbb{R}^{n \times d}$$
$$V \in \mathbb{R}^{d \times k}$$
$$W \in \mathbb{R}^{d \times d}$$

- How do we train this network ?



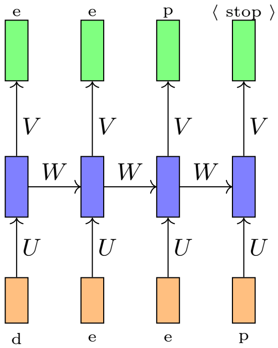


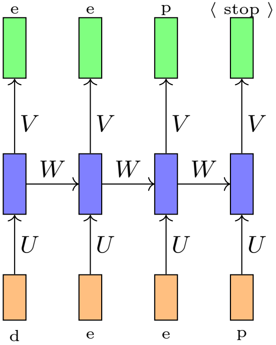
- How do we train this network ? (**Ans**: using backpropagation)



- How do we train this network ? (**Ans**: using backpropagation)
- Let us understand this with a concrete example

- Suppose we consider our task of auto-completion (predicting the next character)





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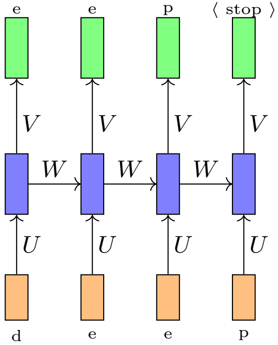

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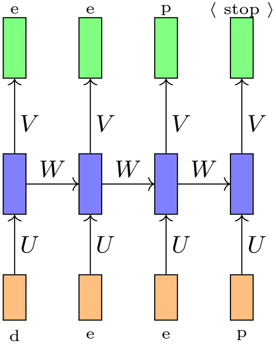

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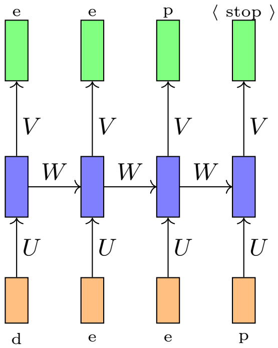

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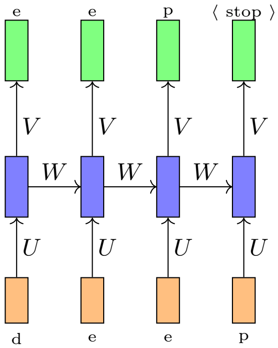

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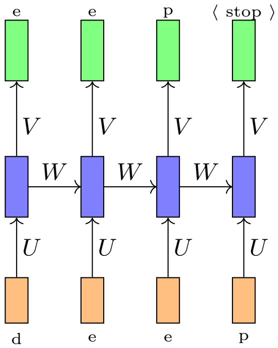

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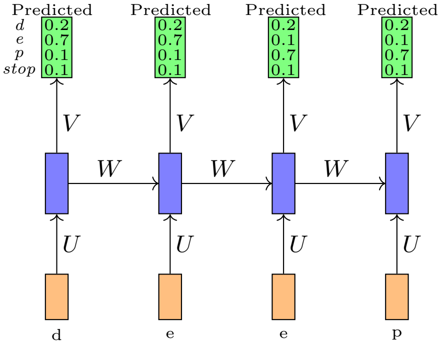
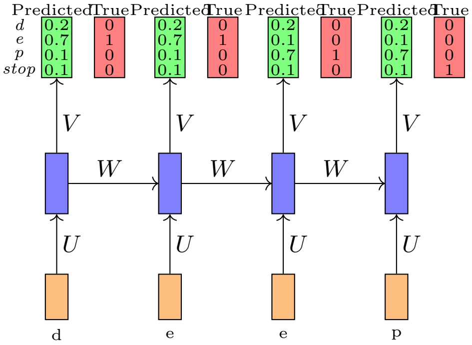


- Suppose we initialize U, V, W randomly and the network predicts the probabilities as shown



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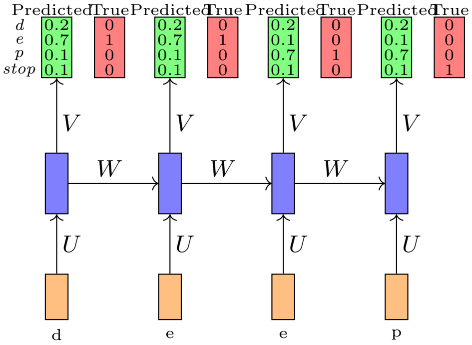

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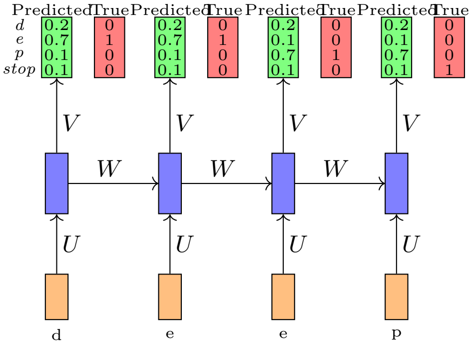

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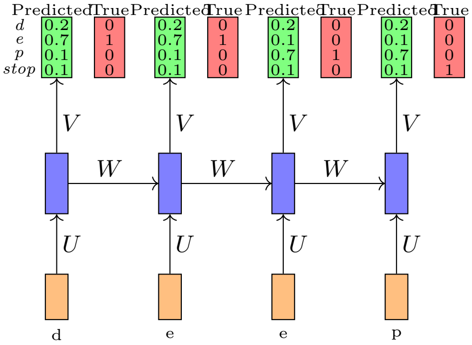

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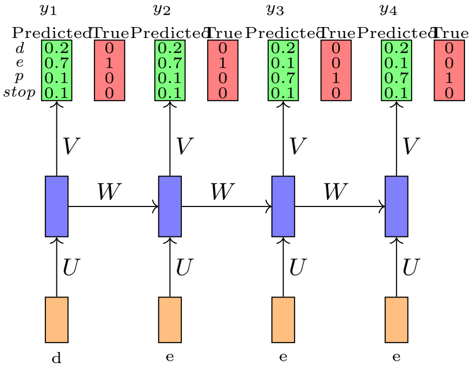
- How do we backpropagate this loss and update the parameters ($\theta = \{U, V, W, b, c\}$) of the network ?



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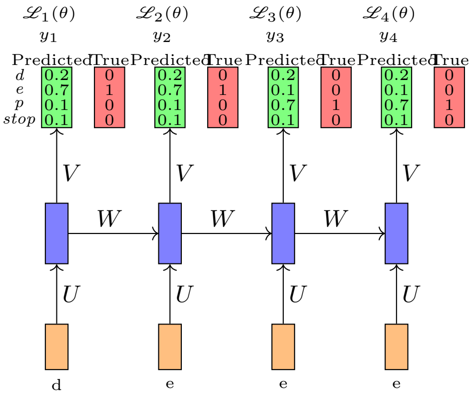

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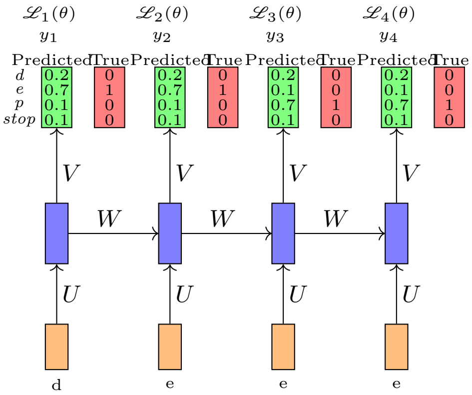

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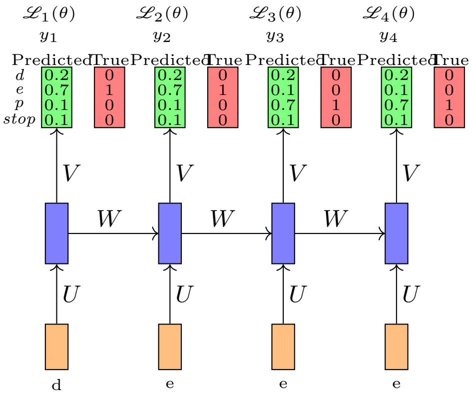

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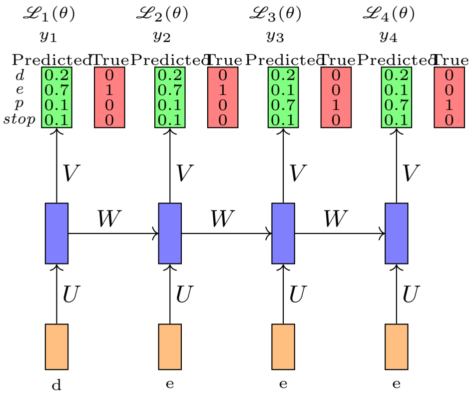

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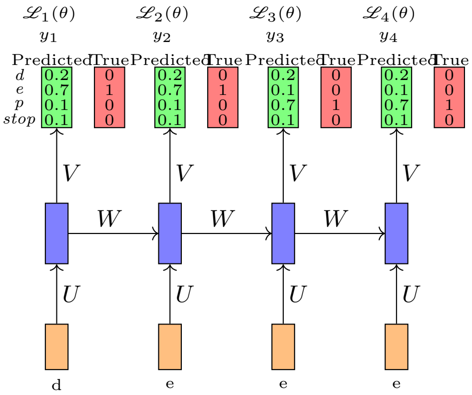

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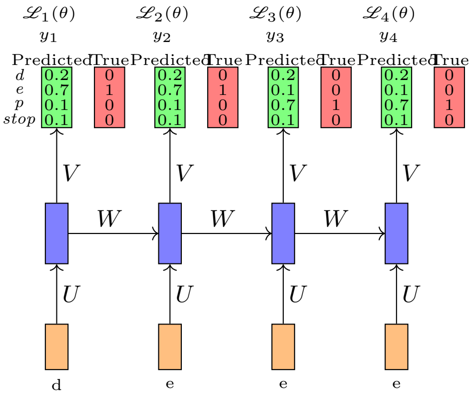

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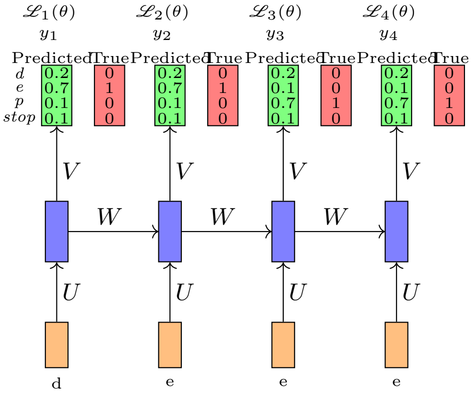

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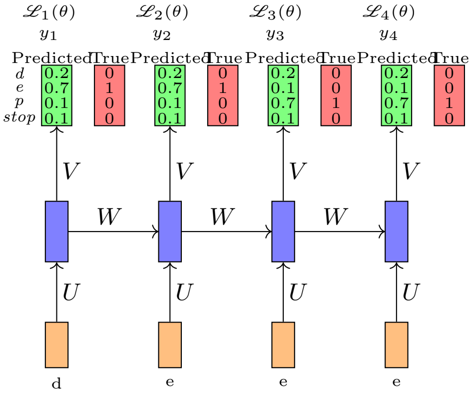

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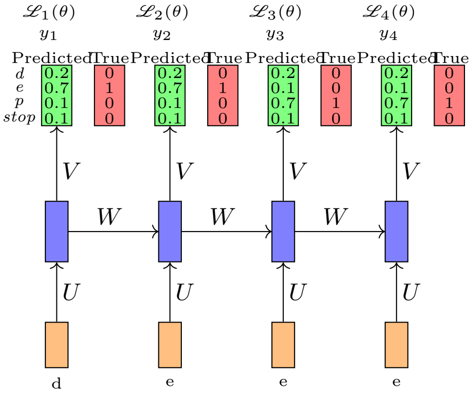

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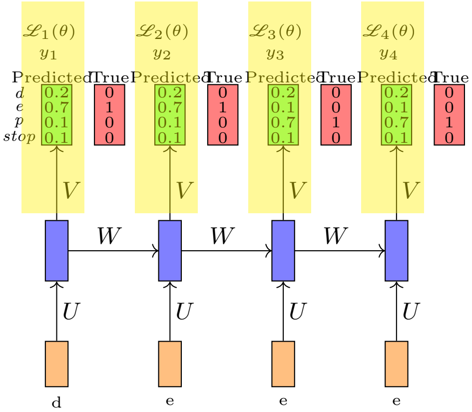

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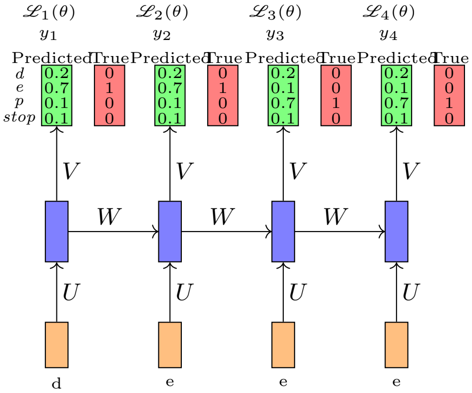

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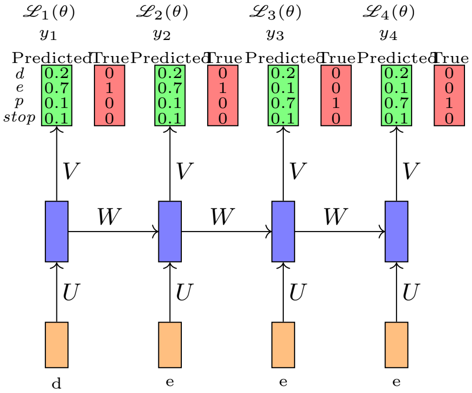

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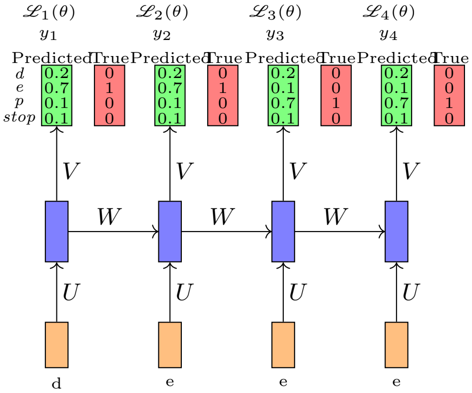

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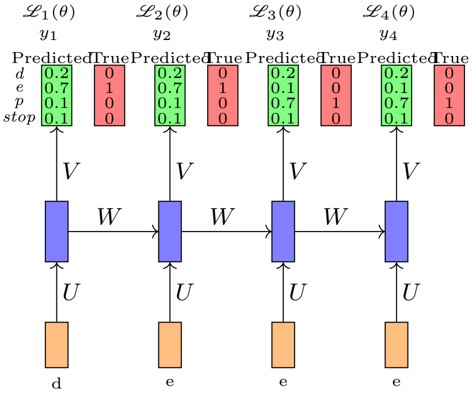

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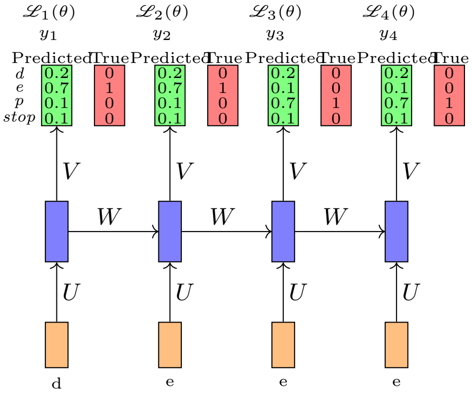

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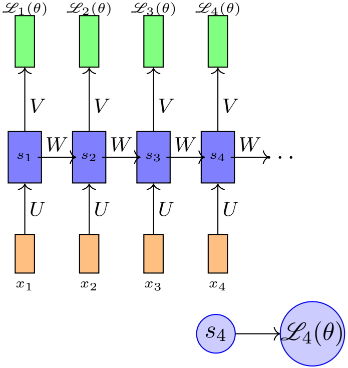
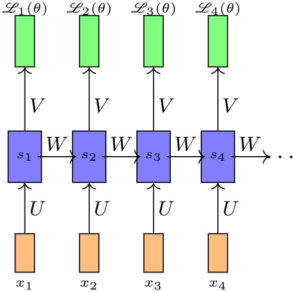
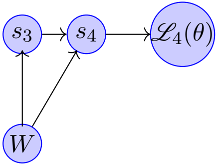
- Let us see this by considering $\mathscr{L}_4(\theta)$

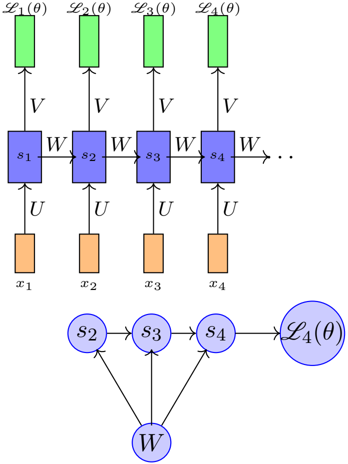
- $\mathscr{L}_4(\theta)$ depends on s_4







- $\mathscr{L}_4(\theta)$ depends on s_4
- s_4 in turn depends on s_3 and W



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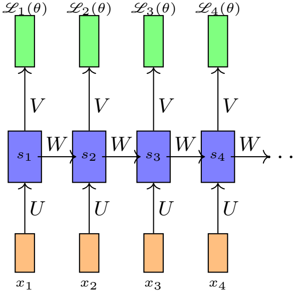

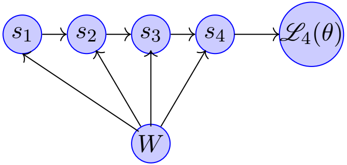

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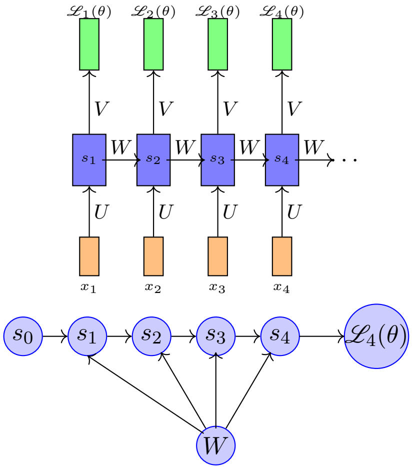

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- s_3 in turn depends on s_2 and W
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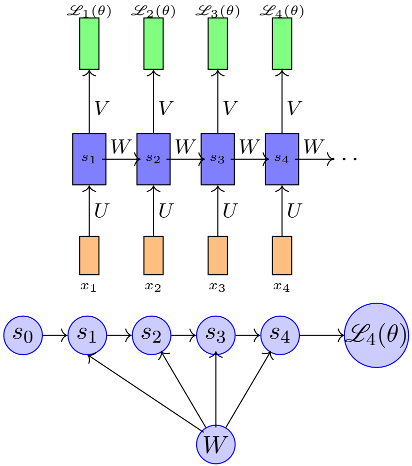
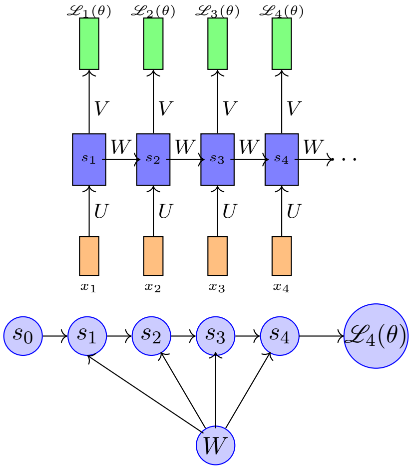
- s_1 in turn depends on s_0 and W

  where s_0 is a constant starting state.



- What we have here is an ordered network



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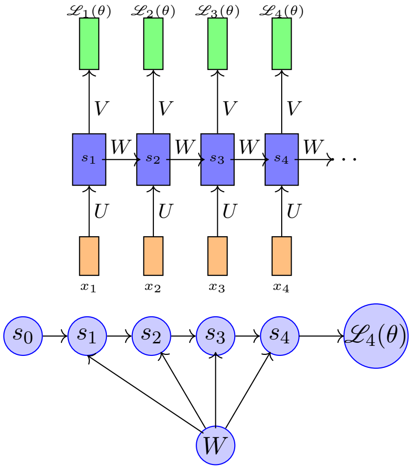

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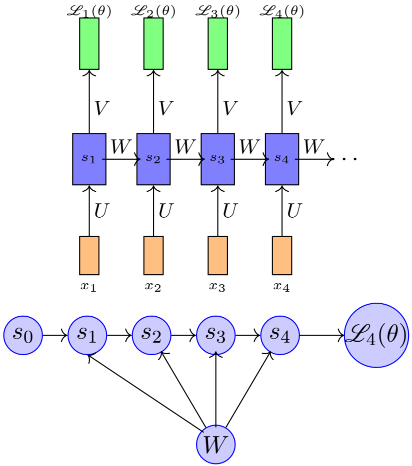

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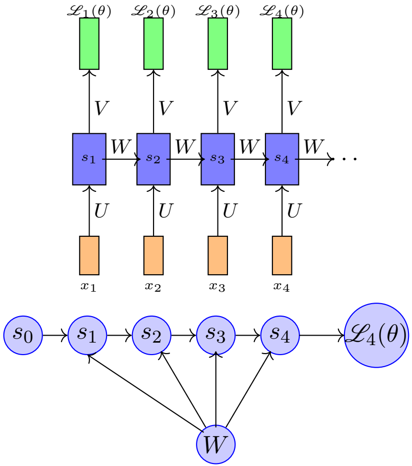

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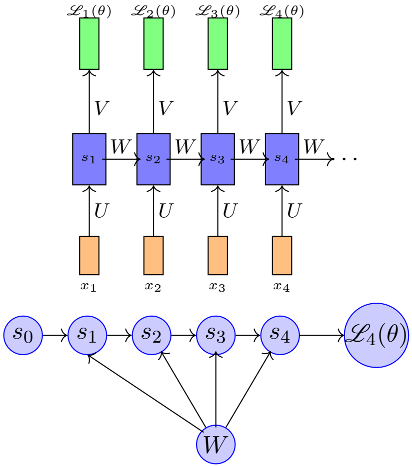
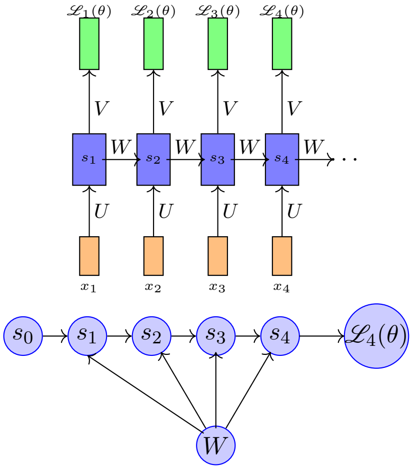
- But how do we compute $\frac{\partial s_4}{\partial W}$



- Recall that

$$s_4 = \sigma(Ws_3 + b)$$



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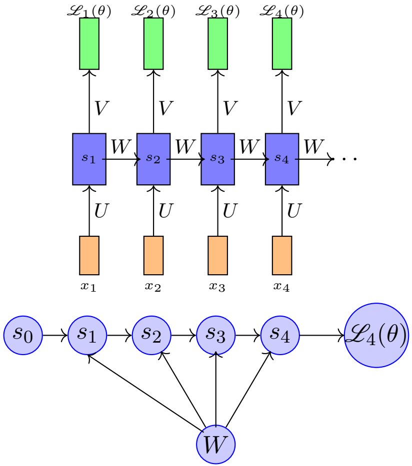

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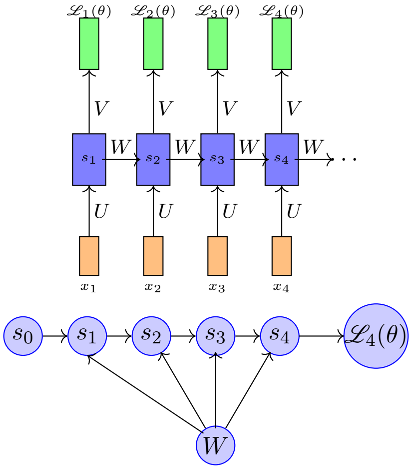

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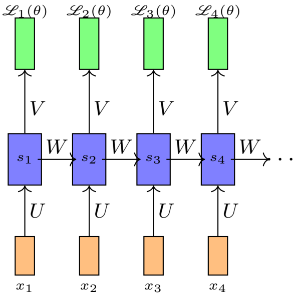

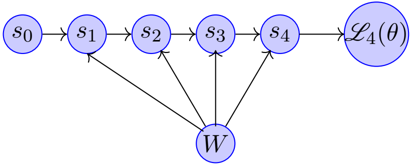

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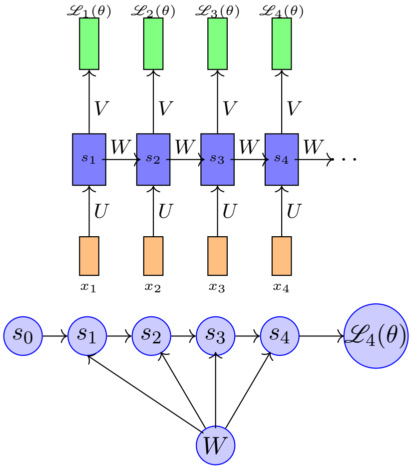
- **Implicit** : Summing over all indirect paths from s_4 to W



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- In such networks the total derivative $\frac{\partial s_4}{\partial W}$ has two parts
- **Explicit** : $\frac{\partial^+ s_4}{\partial W}$, treating all other inputs as constant
- **Implicit** : Summing over all indirect paths from s_4 to W
- Let us see how to do this

$$\frac{\partial s_4}{\partial W} = \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}}$$

$$\frac{\partial s_4}{\partial W} = \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}}$$

$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\left[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\right]$$

$$\frac{\partial s_4}{\partial W} = \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}}$$

$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\Big[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\Big]$$

$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\Big[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\Big]$$

$$
\begin{aligned}
\frac{\partial s_4}{\partial W} &= \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}} \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\Bigg[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\Bigg] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\left[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\right] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial s_1}\left[\frac{\partial^+ s_1}{\partial W}\right]
\end{aligned}
$$

$$\frac{\partial s_4}{\partial W} = \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}}$$

$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\left[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\right]$$

$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\left[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\right]$$

$$= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial s_1}\left[\frac{\partial^+ s_1}{\partial W}\right]$$

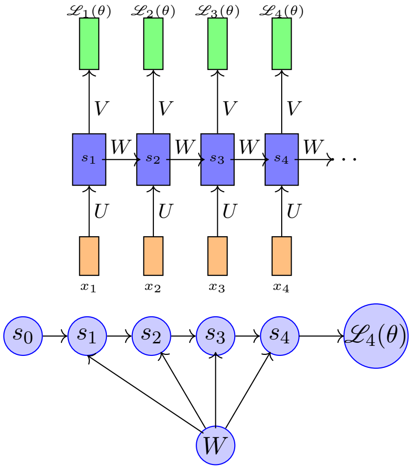
For simplicity we will short-circuit some of the paths

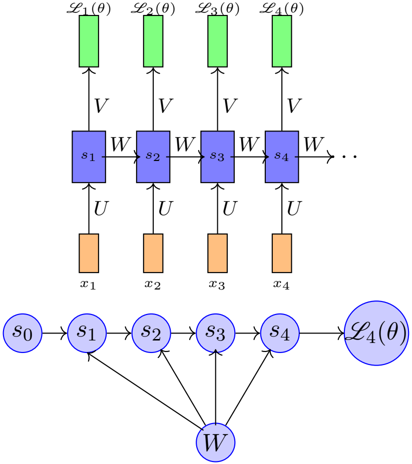
$$
\begin{aligned}
\frac{\partial s_4}{\partial W} &= \underbrace{\frac{\partial^+ s_4}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial W}}_{\text{implicit}} \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\left[\underbrace{\frac{\partial^+ s_3}{\partial W}}_{\text{explicit}} + \underbrace{\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial W}}_{\text{implicit}}\right] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\left[\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial W}\right] \\
&= \frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial s_3}{\partial s_2}\frac{\partial s_2}{\partial s_1}\left[\frac{\partial^+ s_1}{\partial W}\right]
\end{aligned}
$$

For simplicity we will short-circuit some of the paths

$$
\frac{\partial s_4}{\partial W} = \frac{\partial s_4}{\partial s_4}\frac{\partial^+ s_4}{\partial W} + \frac{\partial s_4}{\partial s_3}\frac{\partial^+ s_3}{\partial W} + \frac{\partial s_4}{\partial s_2}\frac{\partial^+ s_2}{\partial W} + \frac{\partial s_4}{\partial s_1}\frac{\partial^+ s_1}{\partial W} = \sum_{k=1}^{4}\frac{\partial s_4}{\partial s_k}\frac{\partial^+ s_k}{\partial W}
$$

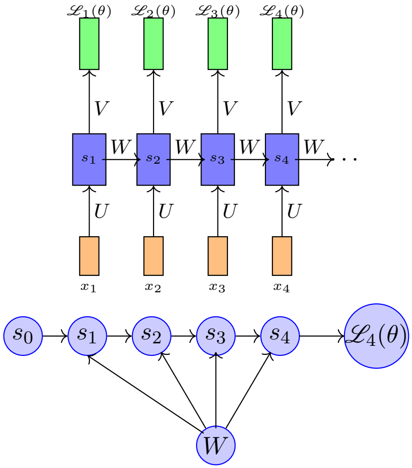
- Finally we have





- Finally we have

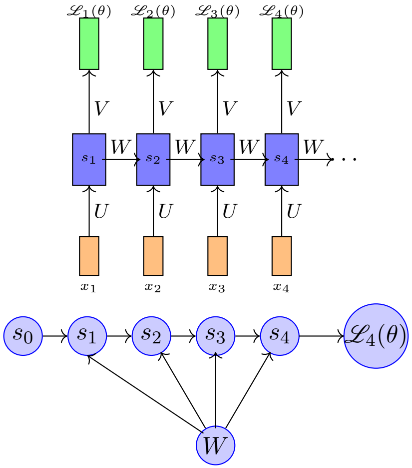
$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4} \frac{\partial s_4}{\partial W}$$



- Finally we have

$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4} \frac{\partial s_4}{\partial W}$$

$$\frac{\partial s_4}{\partial W} = \sum_{k=1}^{4} \frac{\partial s_4}{\partial s_k} \frac{\partial^+ s_k}{\partial W}$$

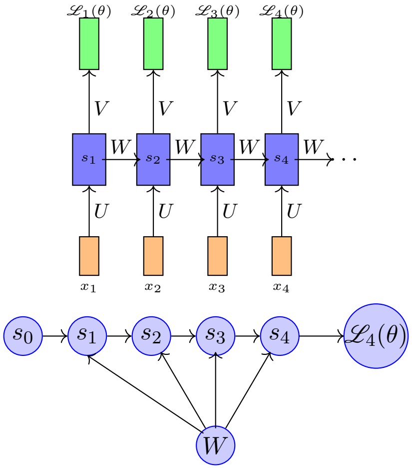


- Finally we have

$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4} \frac{\partial s_4}{\partial W}$$

$$\frac{\partial s_4}{\partial W} = \sum_{k=1}^{4} \frac{\partial s_4}{\partial s_k} \frac{\partial^+ s_k}{\partial W}$$

$$\therefore \frac{\partial \mathscr{L}_t(\theta)}{\partial W} = \frac{\partial \mathscr{L}_t(\theta)}{\partial s_t} \sum_{k=1}^{t} \frac{\partial s_t}{\partial s_k} \frac{\partial^+ s_k}{\partial W}$$



- Finally we have

$$\frac{\partial \mathscr{L}_4(\theta)}{\partial W} = \frac{\partial \mathscr{L}_4(\theta)}{\partial s_4}\frac{\partial s_4}{\partial W}$$

$$\frac{\partial s_4}{\partial W} = \sum_{k=1}^{4}\frac{\partial s_4}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$

$$\therefore \frac{\partial \mathscr{L}_t(\theta)}{\partial W} = \frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}\sum_{k=1}^{t}\frac{\partial s_t}{\partial s_k}\frac{\partial^+ s_k}{\partial W}$$

- This algorithm is called backpropagation through time (BPTT) as we backpropagate over all previous time steps

Module 14.4: The problem of Exploding and Vanishing Gradients

- We will now focus on $\frac{\partial s_t}{\partial s_k}$ and highlight an important problem in training RNN's using BPTT

- We will now focus on $\frac{\partial s_t}{\partial s_k}$ and highlight an important problem in training RNN's using BPTT

$$\frac{\partial s_t}{\partial s_k} = \frac{\partial s_t}{\partial s_{t-1}} \frac{\partial s_{t-1}}{\partial s_{t-2}} \dots \frac{\partial s_{k+1}}{\partial s_k}$$

- We will now focus on $\frac{\partial s_t}{\partial s_k}$ and highlight an important problem in training RNN's using BPTT

$$\frac{\partial s_t}{\partial s_k} = \frac{\partial s_t}{\partial s_{t-1}} \frac{\partial s_{t-1}}{\partial s_{t-2}} \dots \frac{\partial s_{k+1}}{\partial s_k}$$

$$= \prod_{j=k}^{t-1} \frac{\partial s_{j+1}}{\partial s_j}$$

- We will now focus on $\frac{\partial s_t}{\partial s_k}$ and highlight an important problem in training RNN's using BPTT

$$\frac{\partial s_t}{\partial s_k} = \frac{\partial s_t}{\partial s_{t-1}} \frac{\partial s_{t-1}}{\partial s_{t-2}} \cdots \frac{\partial s_{k+1}}{\partial s_k}$$
$$= \prod_{j=k}^{t-1} \frac{\partial s_{j+1}}{\partial s_j}$$

- Let us look at one such term in the product (i.e., $\frac{\partial s_{j+1}}{\partial s_j}$)

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j}\frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & & \\ & & & \\ & & & \end{bmatrix}$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \\ & & & \\ & & & \end{bmatrix}$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ & & & \\ & & & \end{bmatrix}$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ & & & \end{bmatrix}$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j}\frac{\partial a_j}{\partial s_{j-1}}$$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \ldots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \ldots \sigma(a_{jd})]$$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j}\frac{\partial a_j}{\partial s_{j-1}}$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \ldots & \sigma'(a_{jd}) \end{bmatrix}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \dots \sigma(a_{jd})]$$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \dots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & \sigma'(a_{jd}) \end{bmatrix}$$
$$= diag(\sigma'(a_j))$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \dots \sigma(a_{jd})]$$

$$a_j = Ws_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \dots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma^{'}(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma^{'}(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & \sigma^{'}(a_{jd}) \end{bmatrix}$$
$$= diag(\sigma^{'}(a_j))$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$
$$= diag(\sigma^{'}(a_j))W$$

$$a_j = [a_{j1}, a_{j2}, a_{j3}, \dots a_{jd},]$$
$$s_j = [\sigma(a_{j1}), \sigma(a_{j2}), \dots \sigma(a_{jd})]$$

$$\frac{\partial s_j}{\partial a_j} = \begin{bmatrix} \frac{\partial s_{j1}}{\partial a_{j1}} & \frac{\partial s_{j2}}{\partial a_{j1}} & \frac{\partial s_{j3}}{\partial a_{j1}} & \cdots \\ \frac{\partial s_{j1}}{\partial a_{j2}} & \frac{\partial s_{j2}}{\partial a_{j2}} & \ddots & \\ \vdots & \vdots & \vdots & \frac{\partial s_{jd}}{\partial a_{jd}} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma'(a_{j1}) & 0 & 0 & 0 \\ 0 & \sigma'(a_{j2}) & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & \sigma'(a_{jd}) \end{bmatrix}$$
$$= diag(\sigma'(a_j))$$

- We are interested in $\frac{\partial s_j}{\partial s_{j-1}}$

$$a_j = W s_j + b$$
$$s_j = \sigma(a_j)$$

$$\frac{\partial s_j}{\partial s_{j-1}} = \frac{\partial s_j}{\partial a_j} \frac{\partial a_j}{\partial s_{j-1}}$$
$$= diag(\sigma'(a_j))W$$

- We are interested in the magnitude of $\frac{\partial s_j}{\partial s_{j-1}} \leftarrow$ if it is small (large) $\frac{\partial s_t}{\partial s_k}$ and hence $\frac{\partial \mathscr{L}_t}{\partial W}$ will vanish (explode)

$$\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| = \left\| diag(\sigma'(a_j))W \right\|$$

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma^{'}(a_j))W\right\|$$
$$\leq \left\|diag(\sigma^{'}(a_j)\right\| \|W\|$$

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma^{'}(a_j))W\right\|$$
$$\leq \left\|diag(\sigma^{'}(a_j)\right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| = \left\|diag(\sigma^{'}(a_j))W\right\|$$
$$\leq \left\|diag(\sigma^{'}(a_j)\right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\sigma^{'}(a_j) \leq \frac{1}{4} = \gamma \,[\text{if } \sigma \text{ is logistic }]$$

$$\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| = \left\| diag(\sigma^{'}(a_j))W \right\|$$
$$\leq \left\| diag(\sigma^{'}(a_j) \right\| \|W\|$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\sigma^{'}(a_j) \leq \frac{1}{4} = \gamma \, [\text{if } \sigma \text{ is logistic }]$$
$$\leq 1 = \gamma \, [\text{if } \sigma \text{ is tanh }]$$

$$
\begin{aligned}
\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| &= \left\| diag(\sigma^{'}(a_j))W \right\| \\
&\leq \left\| diag(\sigma^{'}(a_j) \right\| \|W\|
\end{aligned}
$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$
\begin{aligned}
\sigma^{'}(a_j) &\leq \frac{1}{4} = \gamma \, [\text{if } \sigma \text{ is logistic }] \\
&\leq 1 = \gamma \, [\text{if } \sigma \text{ is tanh }] \\
\left\| \frac{\partial s_j}{\partial s_{j-1}} \right\| &\leq \gamma \, \|W\| \\
&\leq \gamma\lambda
\end{aligned}
$$

$$\left\|\frac{\partial s_t}{\partial s_k}\right\| = \left\|\prod_{j=k+1}^{t} \frac{\partial s_j}{\partial s_{j-1}}\right\|$$

$$\begin{aligned}\left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| &= \left\|diag(\sigma^{'}(a_j))W\right\| \\ &\leq \left\|diag(\sigma^{'}(a_j)\right\| \left\|W\right\|\end{aligned}$$

$\because \sigma(a_j)$ is a bounded function (sigmoid, tanh) $\sigma^{'}(a_j)$ is bounded

$$\begin{aligned}\sigma^{'}(a_j) &\leq \frac{1}{4} = \gamma \,[\text{if } \sigma \text{ is logistic }] \\ &\leq 1 = \gamma \,[\text{if } \sigma \text{ is tanh }] \\ \left\|\frac{\partial s_j}{\partial s_{j-1}}\right\| &\leq \gamma \left\|W\right\| \\ &\leq \gamma\lambda\end{aligned}$$

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- If $\gamma\lambda < 1$ the gradient will vanish
- If $\gamma\lambda > 1$ the gradient could explode

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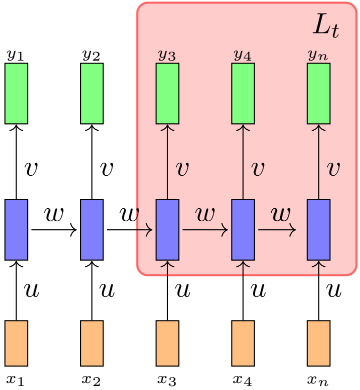
- If $\gamma\lambda < 1$ the gradient will vanish
- If $\gamma\lambda > 1$ the gradient could explode
- This is known as the problem of vanishing/ exploding gradients



- One simple way of avoiding this is to use truncated backpropogation where we restrict the product to $\tau(< t - k)$ terms

Module 14.5: Some Gory Details

$$\underbrace{\frac{\partial \mathscr{L}_t(\theta)}{\partial W}} = \underbrace{\frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}} \sum_{k=1}^{t} \underbrace{\frac{\partial s_t}{\partial s_k}} \quad \underbrace{\frac{\partial^+ s_k}{\partial W}}$$

$$\underbrace{\frac{\partial \mathscr{L}_t(\theta)}{\partial W}}_{\in \mathbb{R}^{d\times d}} = \underbrace{\frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}} \sum_{k=1}^{t} \underbrace{\frac{\partial s_t}{\partial s_k}} \quad \underbrace{\frac{\partial^+ s_k}{\partial W}}$$

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- We know how to compute $\frac{\partial \mathscr{L}_t(\theta)}{\partial s_t}$ (derivative of $\mathscr{L}_t(\theta)$ (scalar) w.r.t. last hidden layer (vector)) using backpropagation

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- We just saw a formula for $\frac{\partial s_t}{\partial s_k}$ which is the derivative of a vector w.r.t. a vector)

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- $\frac{\partial^+ s_k}{\partial W}$ is a tensor $\in \mathbb{R}^{d \times d \times d}$, the derivative of a vector $\in \mathbb{R}^d$ w.r.t. a matrix $\in \mathbb{R}^{d \times d}$

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- How do we compute $\frac{\partial^+ s_k}{\partial W}$? Let us see

- We just look at one element of this $\frac{\partial^+ s_k}{\partial W}$ tensor

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- $\frac{\partial^+ s_{kp}}{\partial W_{qr}}$ is the (p, q, r)-th element of the 3d tensor

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  $a_k = W s_{k-1} + b$

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- $\frac{\partial^+ s_{kp}}{\partial W_{qr}}$ is the (p, q, r)-th element of the 3d tensor

  $a_k = W s_{k-1} + b$

  $s_k = \sigma(a_k)$

$$a_k = Ws_{k-1}$$

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$$\begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kp} \\ \vdots \\ a_{kd} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} & \dots & W_{1d} \\ \\ \vdots & \vdots & \vdots & \vdots \\ W_{p1} & W_{p2} & \dots & W_{pd} \\ \vdots & \vdots & \vdots & \vdots \\ \\ \end{bmatrix} \begin{bmatrix} s_{k-1,1} \\ s_{k-1,2} \\ \vdots \\ s_{k-1,p} \\ \vdots \\ s_{k-1,d} \end{bmatrix}$$

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$$a_{kp} = \sum_{i=1}^{d} W_{pi} s_{k-1,i}$$

$$a_k = Ws_{k-1}$$

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$$\frac{\partial s_{kp}}{\partial W_{qr}} = \frac{\partial s_{kp}}{\partial a_{kp}} \frac{\partial a_{kp}}{\partial W_{qr}}$$

$$
\begin{aligned}
a_k &= W s_{k-1} \\
\begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kp} \\ \vdots \\ a_{kd} \end{bmatrix} &= \begin{bmatrix} W_{11} & W_{12} & \dots & W_{1d} \\ \vdots & \vdots & \vdots & \vdots \\ W_{p1} & W_{p2} & \dots & W_{pd} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} s_{k-1,1} \\ s_{k-1,2} \\ \vdots \\ s_{k-1,p} \\ \vdots \\ s_{k-1,d} \end{bmatrix} \\
a_{kp} &= \sum_{i=1}^{d} W_{pi} s_{k-1,i} \\
s_{kp} &= \sigma(a_{kp}) \\
\frac{\partial s_{kp}}{\partial W_{qr}} &= \frac{\partial s_{kp}}{\partial a_{kp}} \frac{\partial a_{kp}}{\partial W_{qr}} \\
&= \sigma'(a_{kp}) \frac{\partial a_{kp}}{\partial W_{qr}}
\end{aligned}
$$

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