CS 6130 : Advanced Graph Algorithms

Tutte's Theorem and its Proof.

Tutte's Theorem : characterizing graphs which admit a perfect matching

G admits a perfect matching iff
$\forall S \subseteq V \quad o(G \setminus S) \leq |S|$.

(Easy direction) If G admits a perfect matching then $\forall S \subseteq V \quad o(G \setminus S) \leq |S|$.

Some terminology : If S s.t $o(G \setminus S) \leq |S|$ → good set else bad set.

Tutte's Theorem : characterizing graphs which admit a perfect matching

To prove : If every set is "good" then G admits a perfect matching. (PM).

Assume for contradiction that above is false.

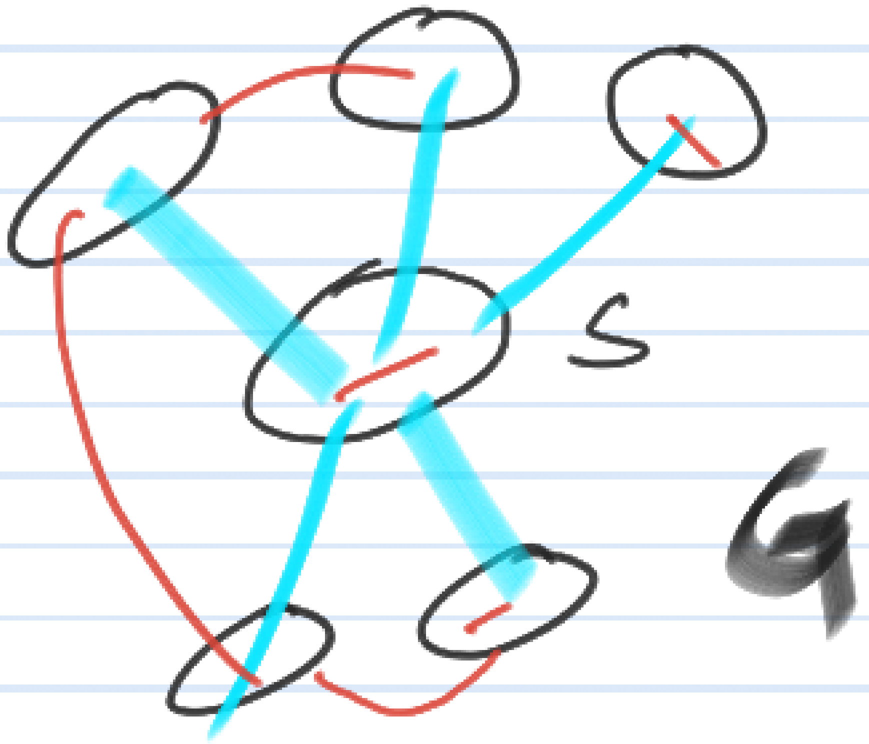
$\Rightarrow \exists\, G$ s.t every $S \subseteq V$ is "good" and G does not admit a PM

Edge maximal graph

By assumption : $\exists G$: every $S \subseteq V$ is good and
earlier G does not admit PM

Consider $G + (u,v)$ s.t $(u\,v) \notin E$

What properties does G satisfy ?



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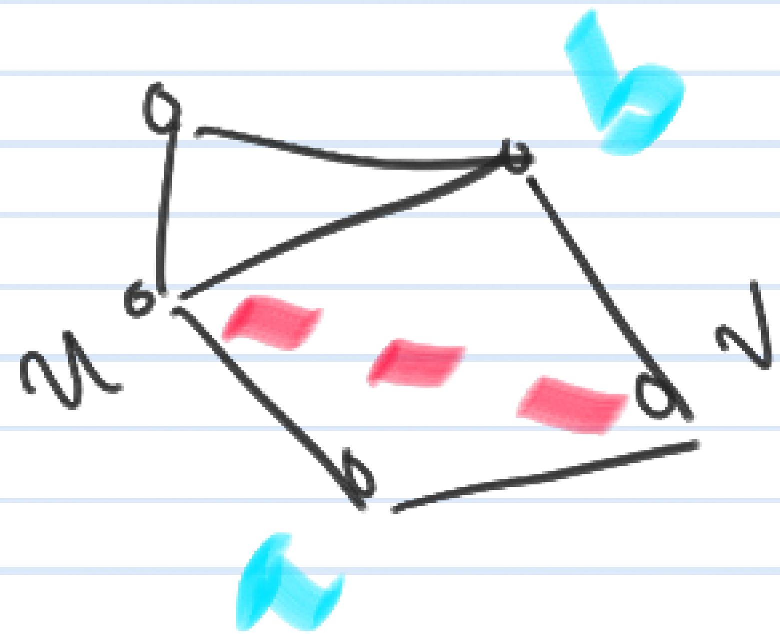
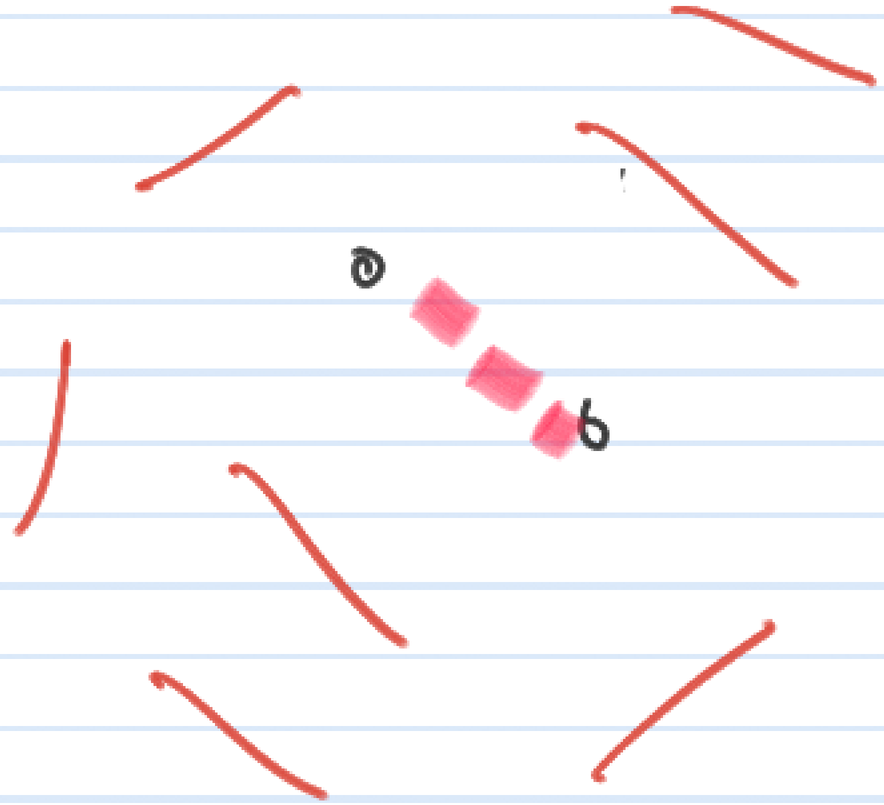
$$G \xrightarrow[(u\,v)]{+} G_1 \xrightarrow[(a,b)]{+} G_2 \xrightarrow{+} \cdots \rightarrow G_k = G'$$

edge maximal graph.





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Properties :

1) G' has no PM

2) for every $e \notin G'$
$G' + e$ admits a PM

3) good set in G
$\Rightarrow$ good set in G'

4) bad set in G'
$\Rightarrow$ bad set in G

edge maximal graph.

is this the case?

Suppose: Edge Maximal graph G' has additional properties

$v : d(v) = n-1$

1) There exists a set X of Universal vertices

2) $G \setminus X$ decomposes into a collection of cliques

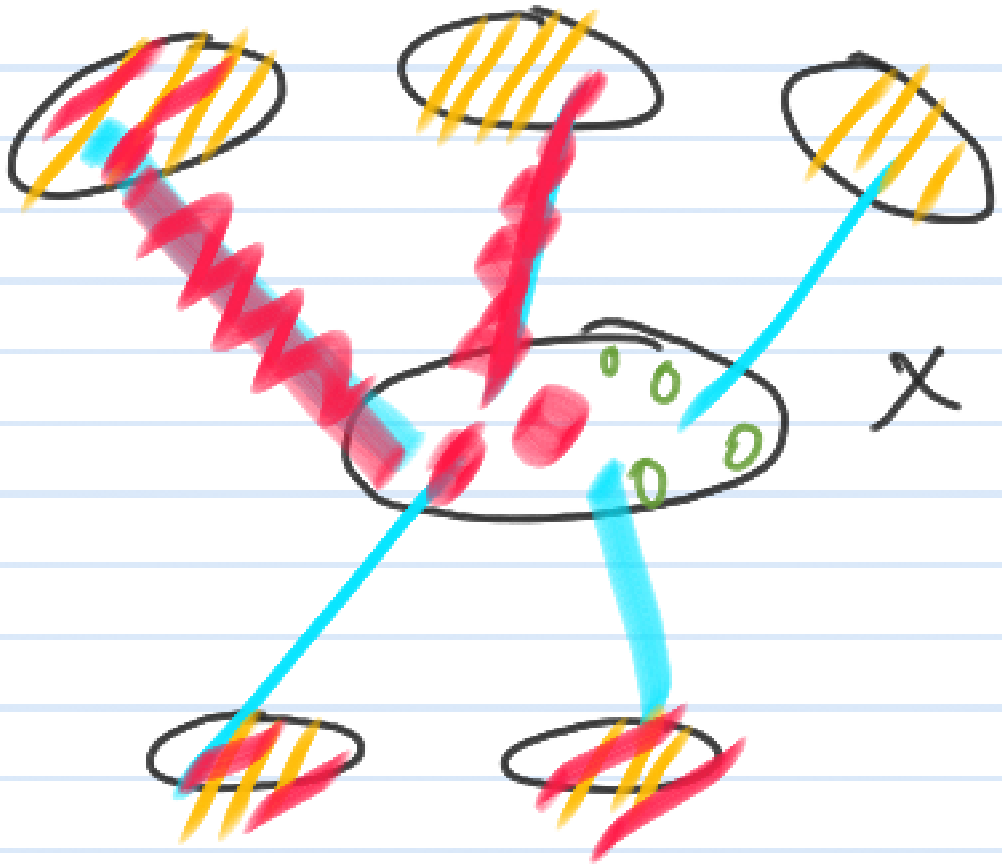


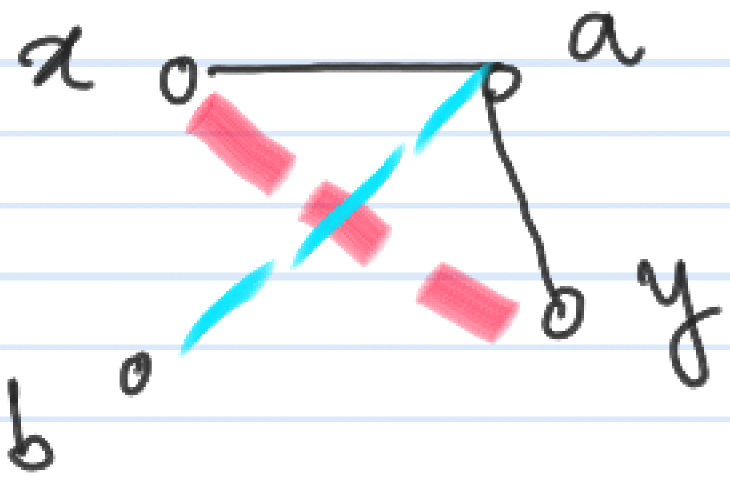
Exhibit a perfect matching.

Suppose : G' does not have <u>additional</u> properties

1) no universal vertices

2) even if X is non empty $G \setminus X$ does not decompose into cliques

We exhibit 4 vertices having following structure



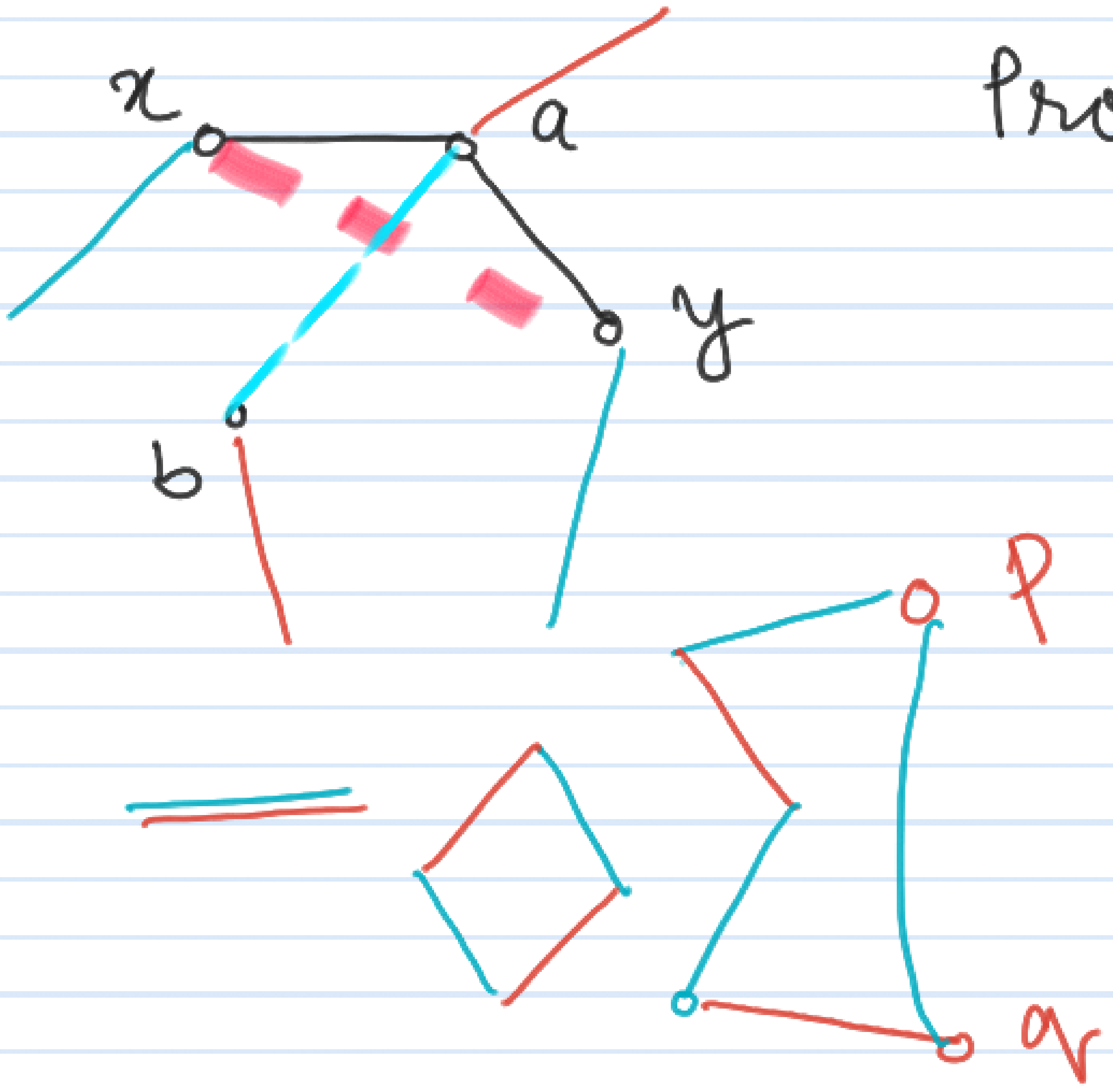
: non edges

① x and y are non adjacent in G'

② x and y have a common neighbour a.

③ a has a non-neighbour b in G'

Case 2: Existence of a special set of vertices



Property of $G + (x, y)$: admits M_1

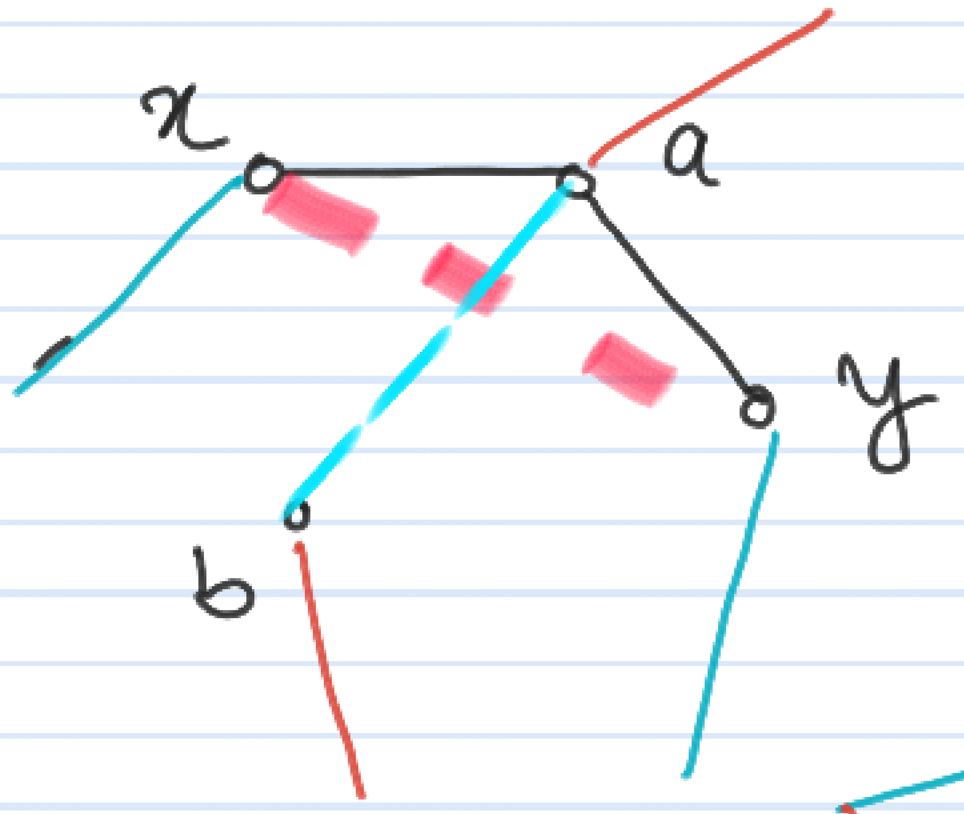
$G + (a, b)$: admits M_2

Let $H = (V, M_1 \oplus M_2)$

$H \cup (ab)(x, y)$: collection of alternating cycles.

x & y belong to some such cycle.

Case 2: Existence of a special set of vertices



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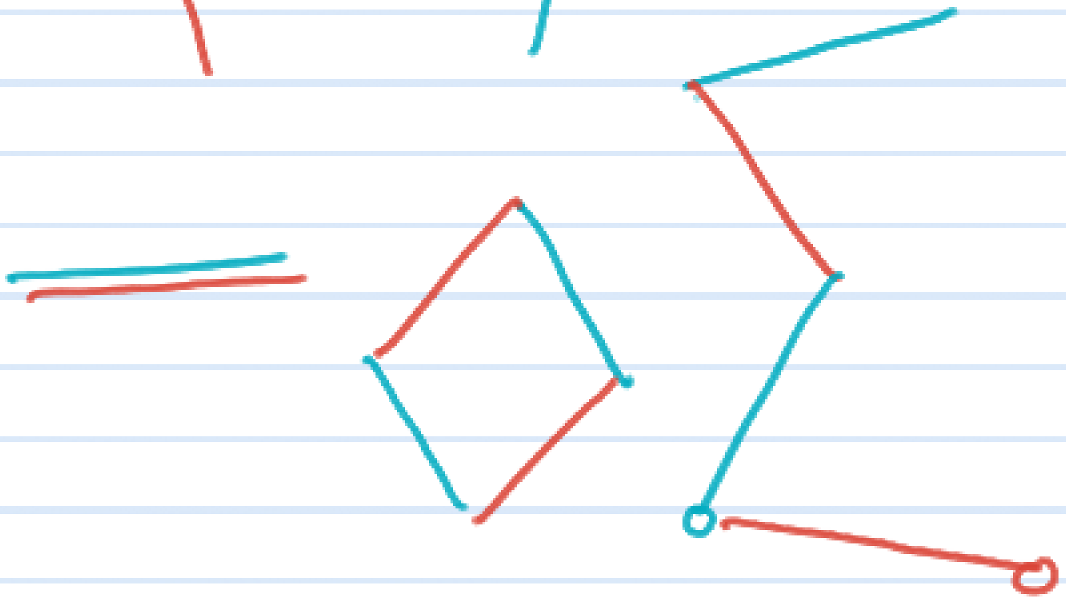
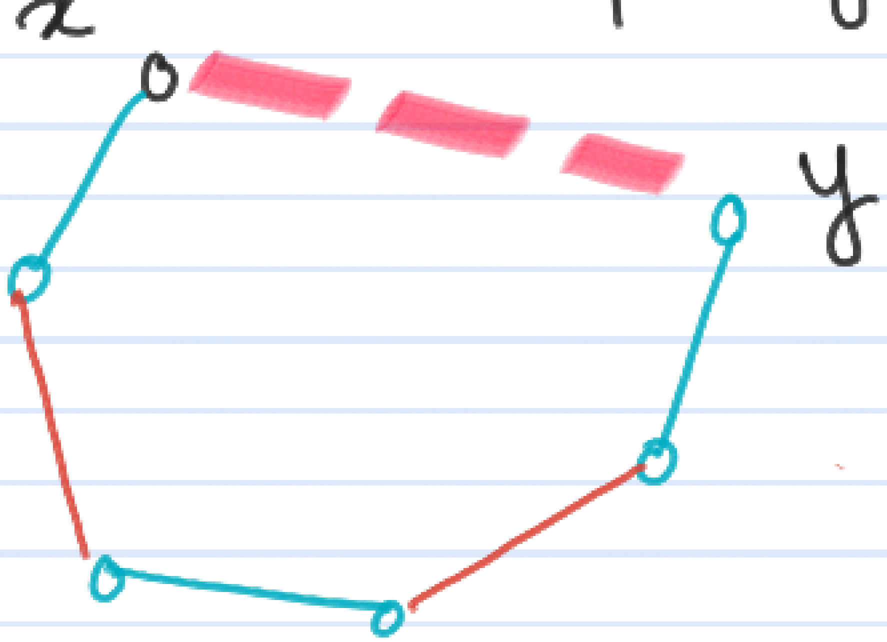


Consider $H' = (V, M_1 \cup M_2 + xa, + ay) + xy + ab$

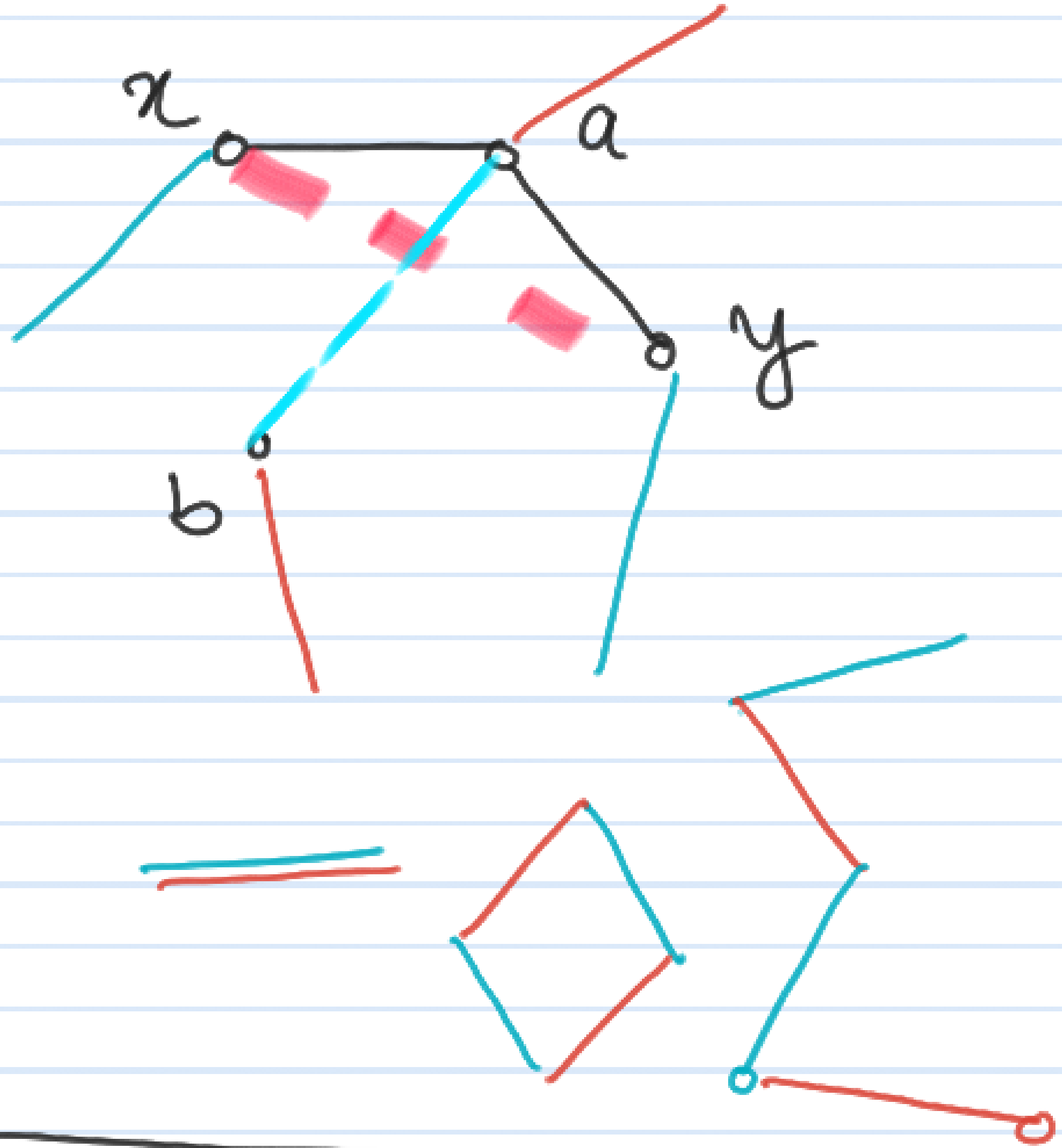


Assume: cycle containing (x, y) does not contain (a, b)

A perfect matching in G'

Blue edges in C, red edges elsewhere

Case 2: Existence of a special set of vertices



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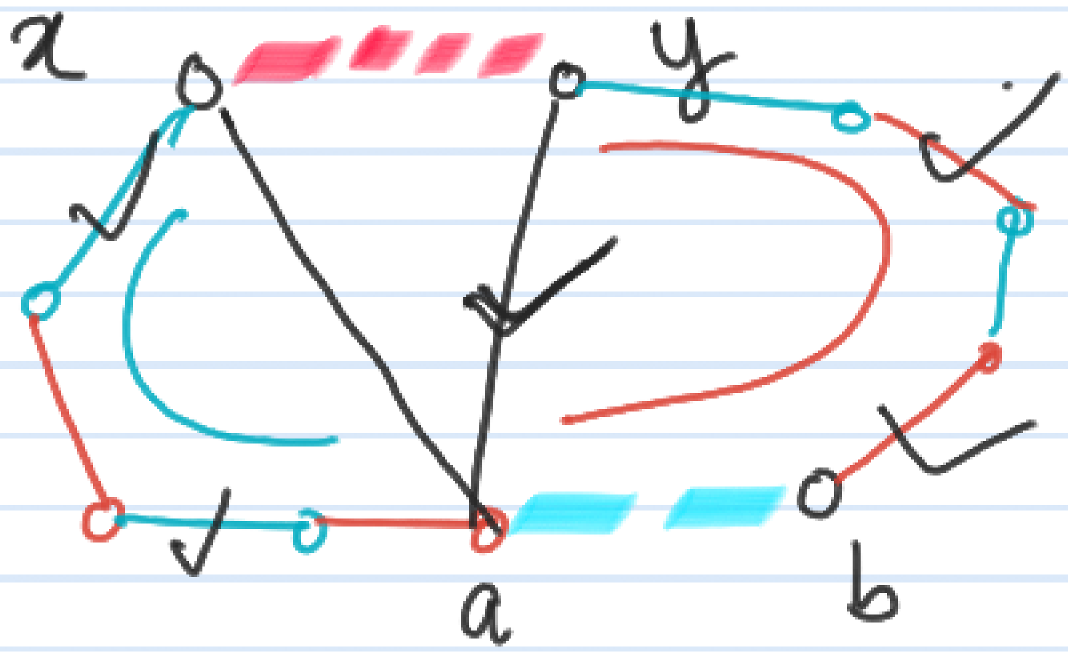

PM in G':

edges of C + ay + red edges elsewhere.

Summary of Proof.

- Defined an edge maximal graph
- A universal set X with properties on $G \setminus X$
- Absence of Universal set or properties on $G \setminus X$
  - → to ensure certain vertices
- Finally show that G' has a PM contradicting the assumption.