CS6130 : Advanced Graph Algorithms

- Matchings in General Graphs
  - Tutte's theorem
  - Gallai Edmond's Decomposition Theorem.

Tutte's Theorem : Characterizing graphs with perfect matching

G : general graph

G admits a perfect matching M.

Let S be any arbitrary subset of V

How does $G \setminus S$ look like ?

Tutte's Theorem : Characterizing graphs with perfect matching

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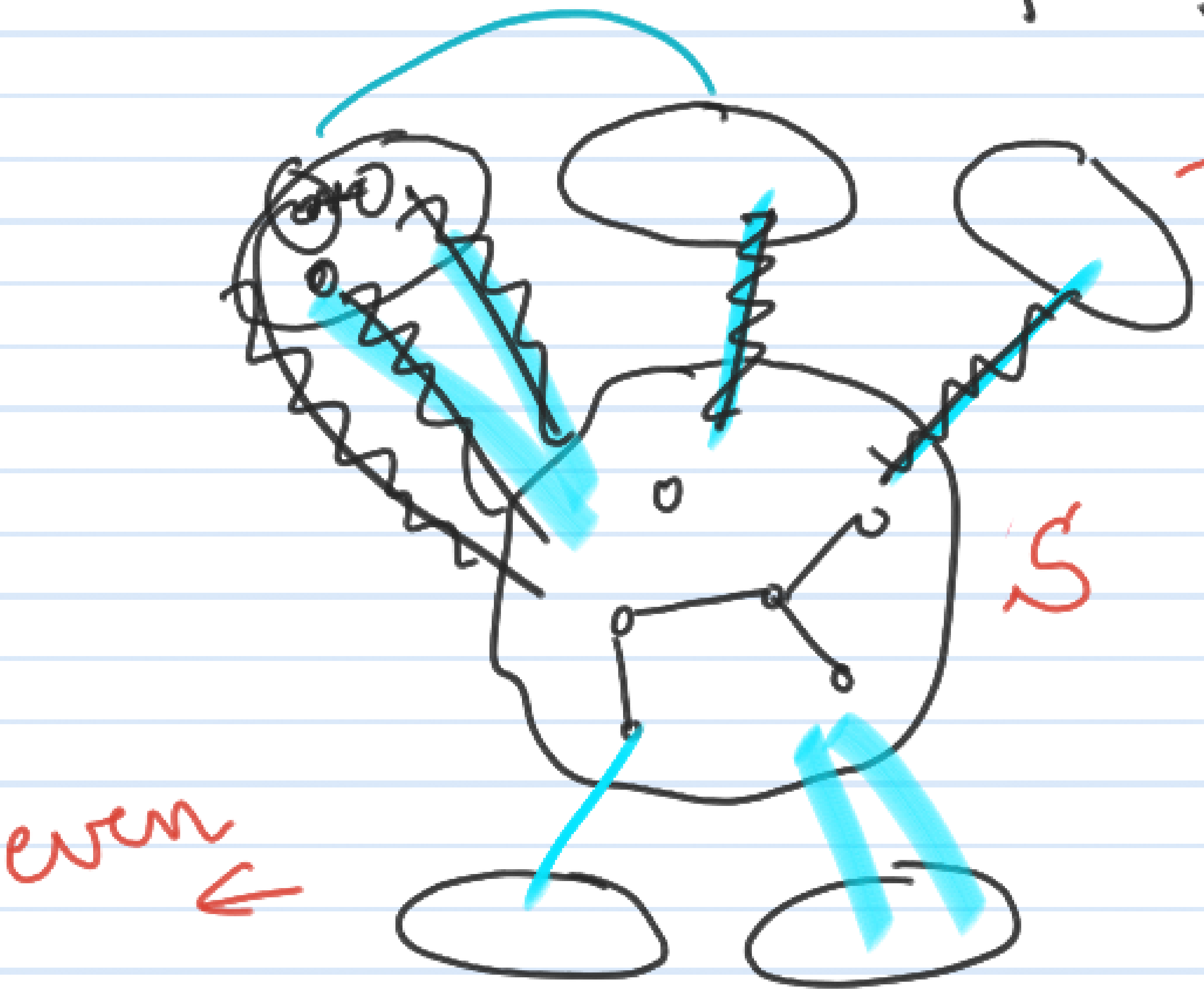

odd component :

even component :

How does M look like in relation to this picture ?

Tutte's Theorem : Characterizing graphs with perfect matching

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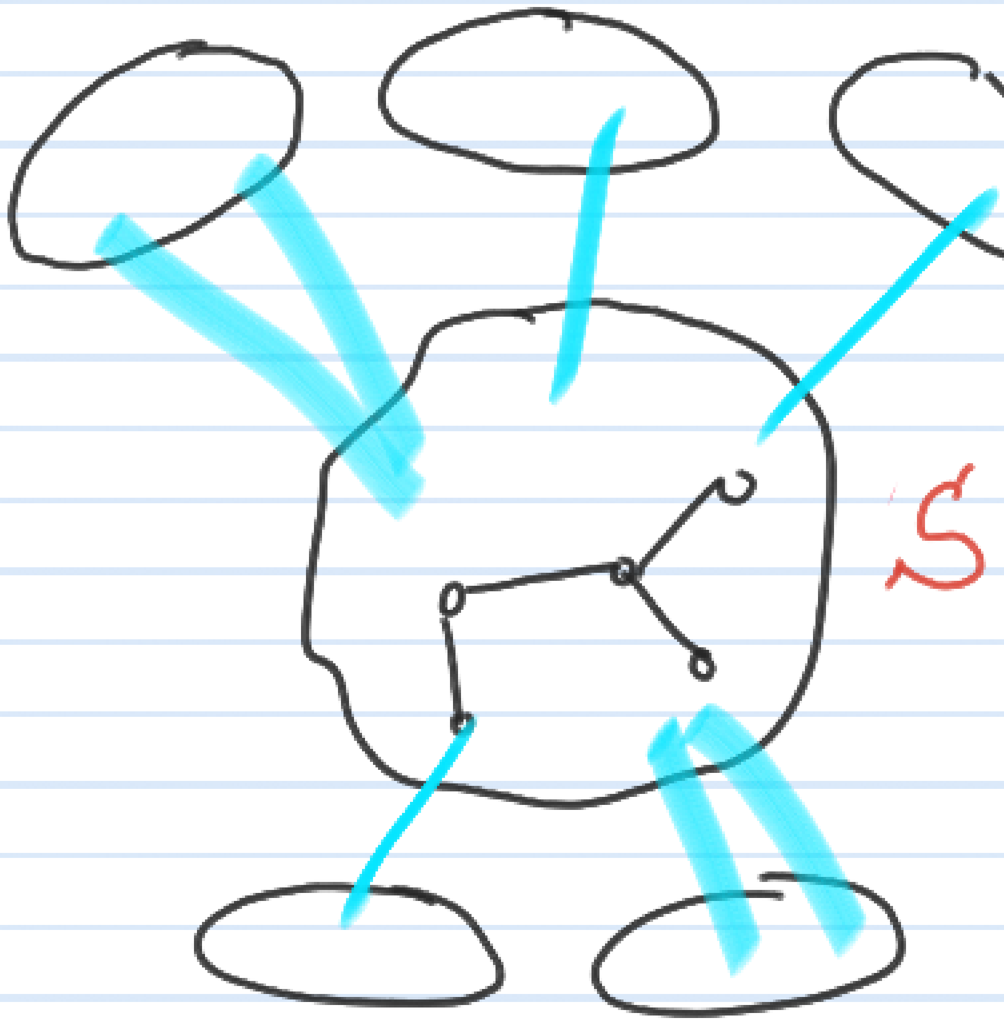
G admits a perfect matching M.

Define $o(G\setminus S)$



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Tutte's Theorem : Characterizing graphs with perfect matching

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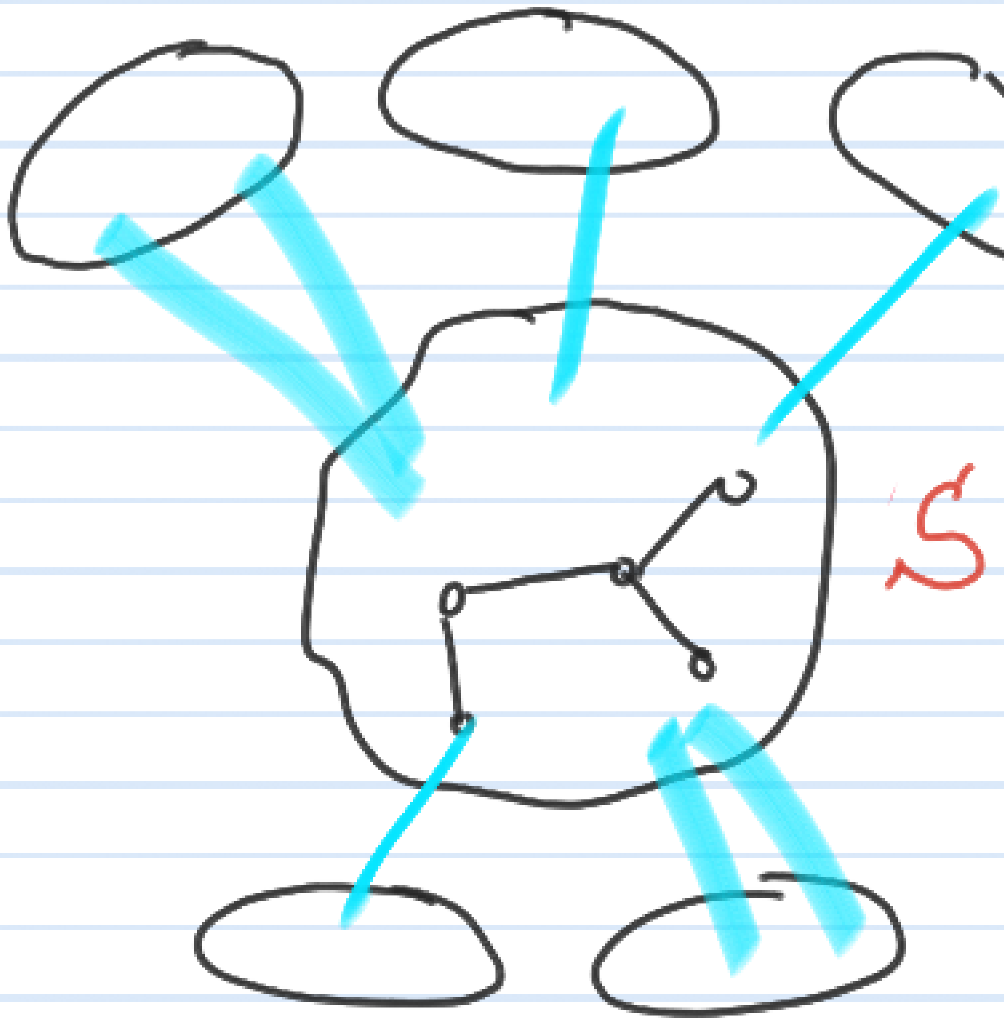


$o(G \setminus S)$ = # of odd sized components in $G \setminus S$

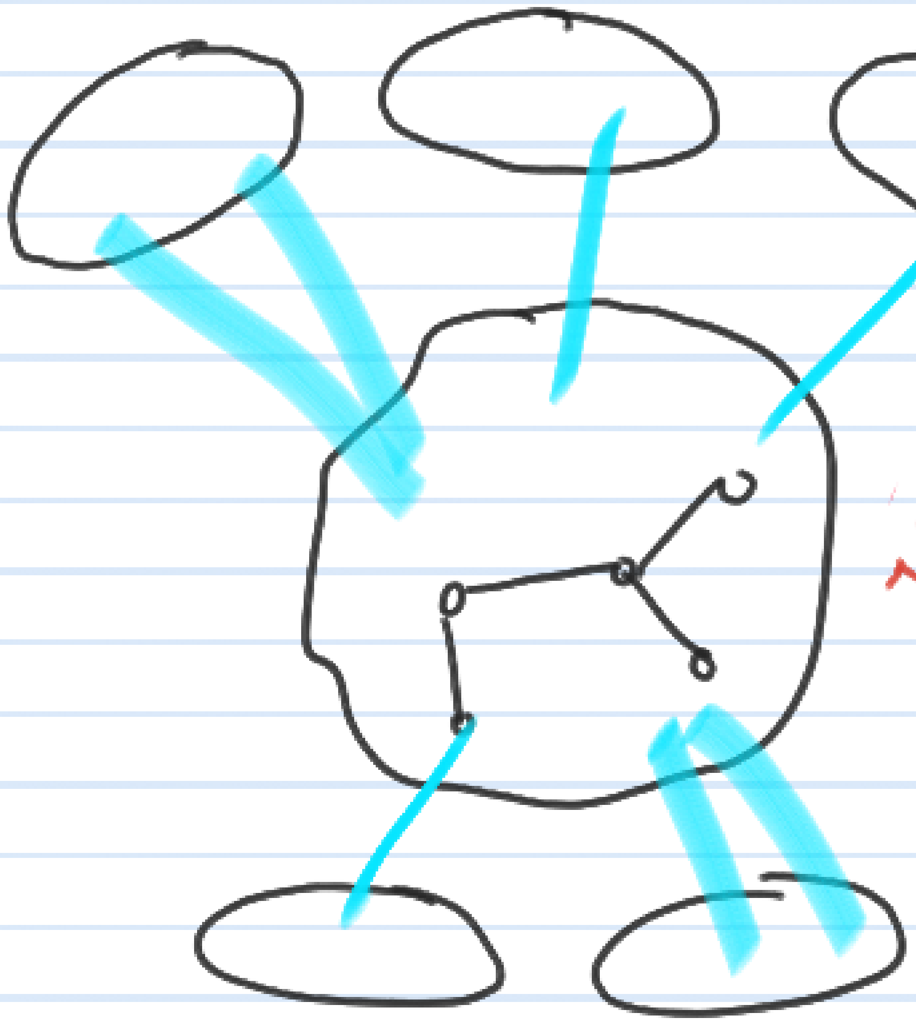
<u>Easy direction</u> : If G admits a perfect matching then $\forall S \subseteq V$:

$o(G \setminus S) \leq |S|$

Tutte's Theorem : Characterizing graphs with perfect matching

G : general graph

G admits a perfect matching M.



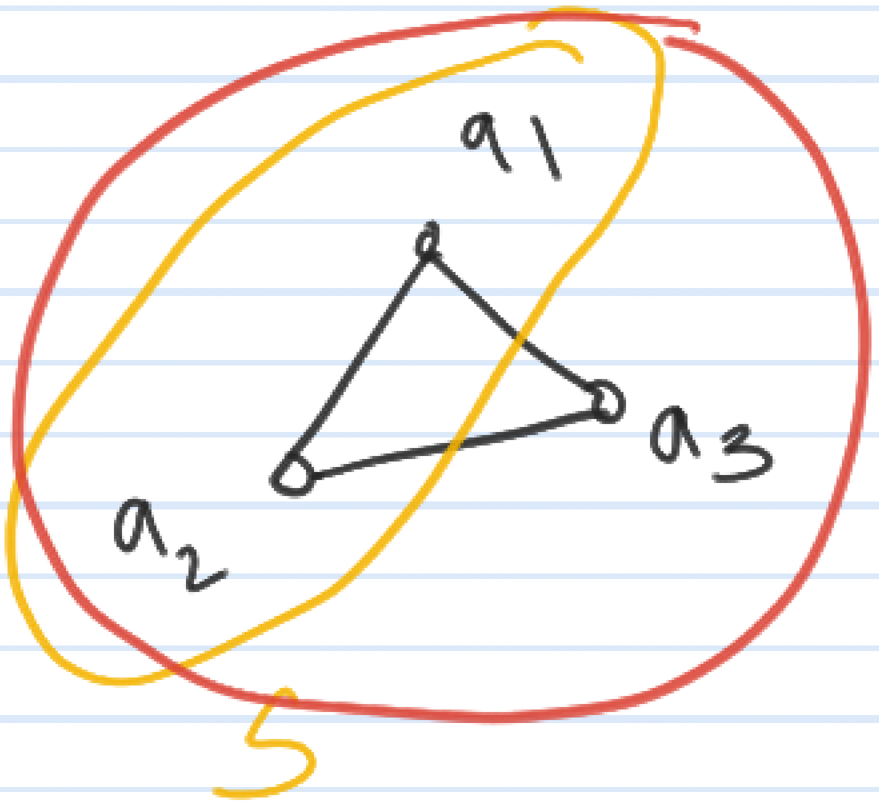
S

$o(G\setminus S)$ = # of odd sized components in $G\setminus S$

Not so

Easy direction : If $\forall S \subseteq V$ $o(G\setminus S) \leq |S|$ then G admits a perfect matching

Example



witness set S s.t

$$\boxed{o(G \setminus S) > |S|}$$

$o(G \setminus S) = 1$

$|S| = 2$

a_3

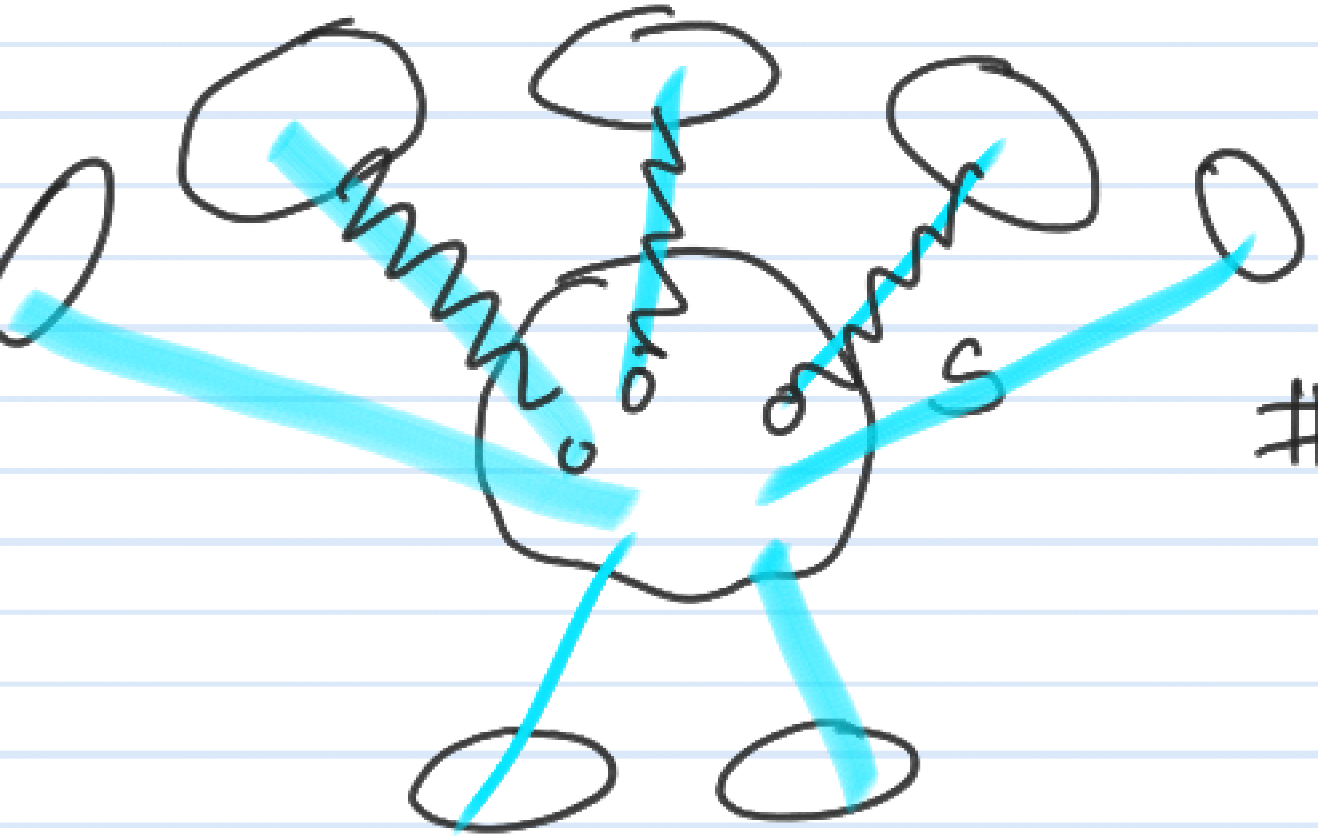
S' $|S'| = 3.$ $o(G \setminus S') = 0$

$S'' = \{\}$ $|S''| = 0;$ $o(G \setminus S'') = 1.$

Tutte's theorem: statement

$$\boxed{\begin{array}{c}\text{A graph } G \text{ admits a perfect matching iff} \\ \forall S \subseteq V \quad o(G \setminus S) \leq |S|.\end{array}}$$

Implication for graphs without a perfect matching?



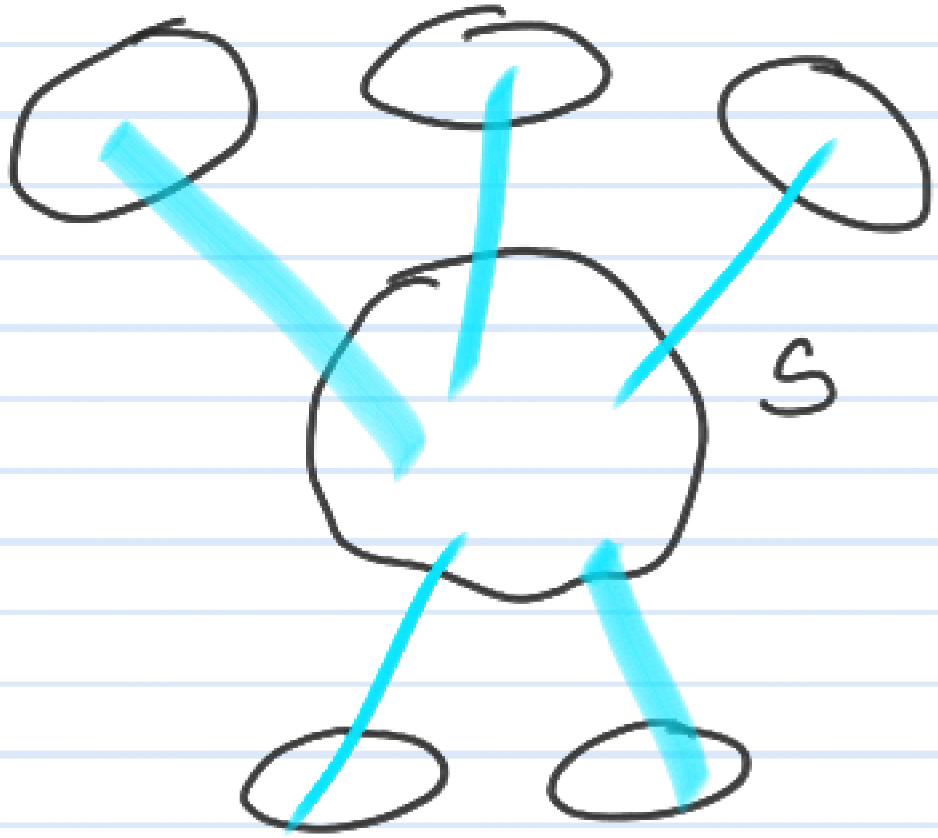
$$\mathrm{def}(S) = o(G \setminus S) - |S|$$

$\#$ of vertices matched $\leq n - \mathrm{def}(S)$

Tutte's theorem: statement

A graph G admits a perfect matching iff
$\forall S \subseteq V \quad o(G \setminus S) \leq |S|.$

Implication for graphs without a perfect matching?



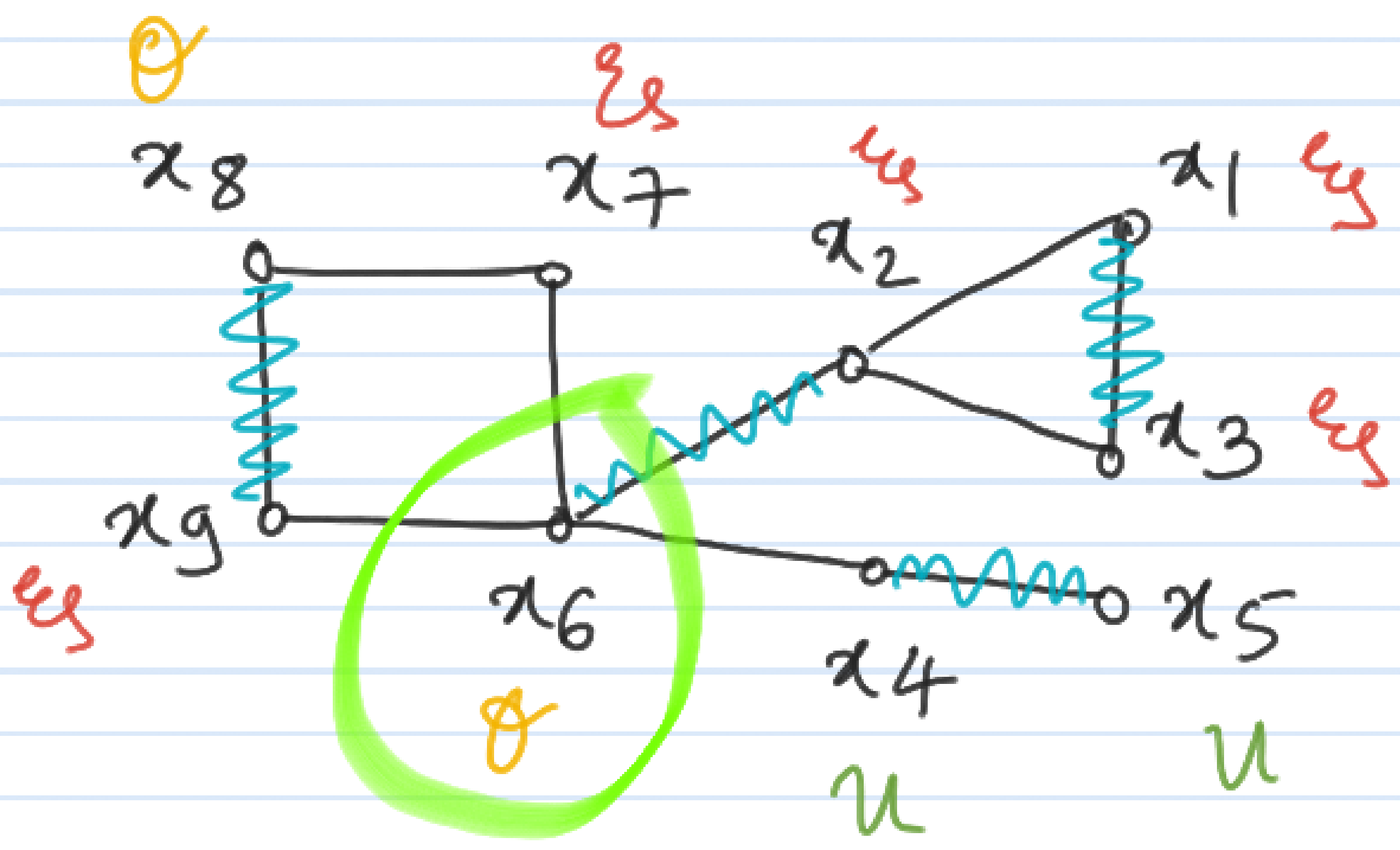
$$def(S) = o(G \setminus S) - |S|$$

$$\# \text{ of vertices matched } \leq n - def(S)$$

To convince that M is maximum construct S such that:

$$|M| = \frac{1}{2}\left[n - def(S)\right]$$

Back to Edmond's Algo : Optimality Certificate



1. Compute max M

2. Label vertices as $\mathcal{O}$, $\mathcal{E}_s$, $\mathcal{U}$.

$\mathcal{E}_s$:

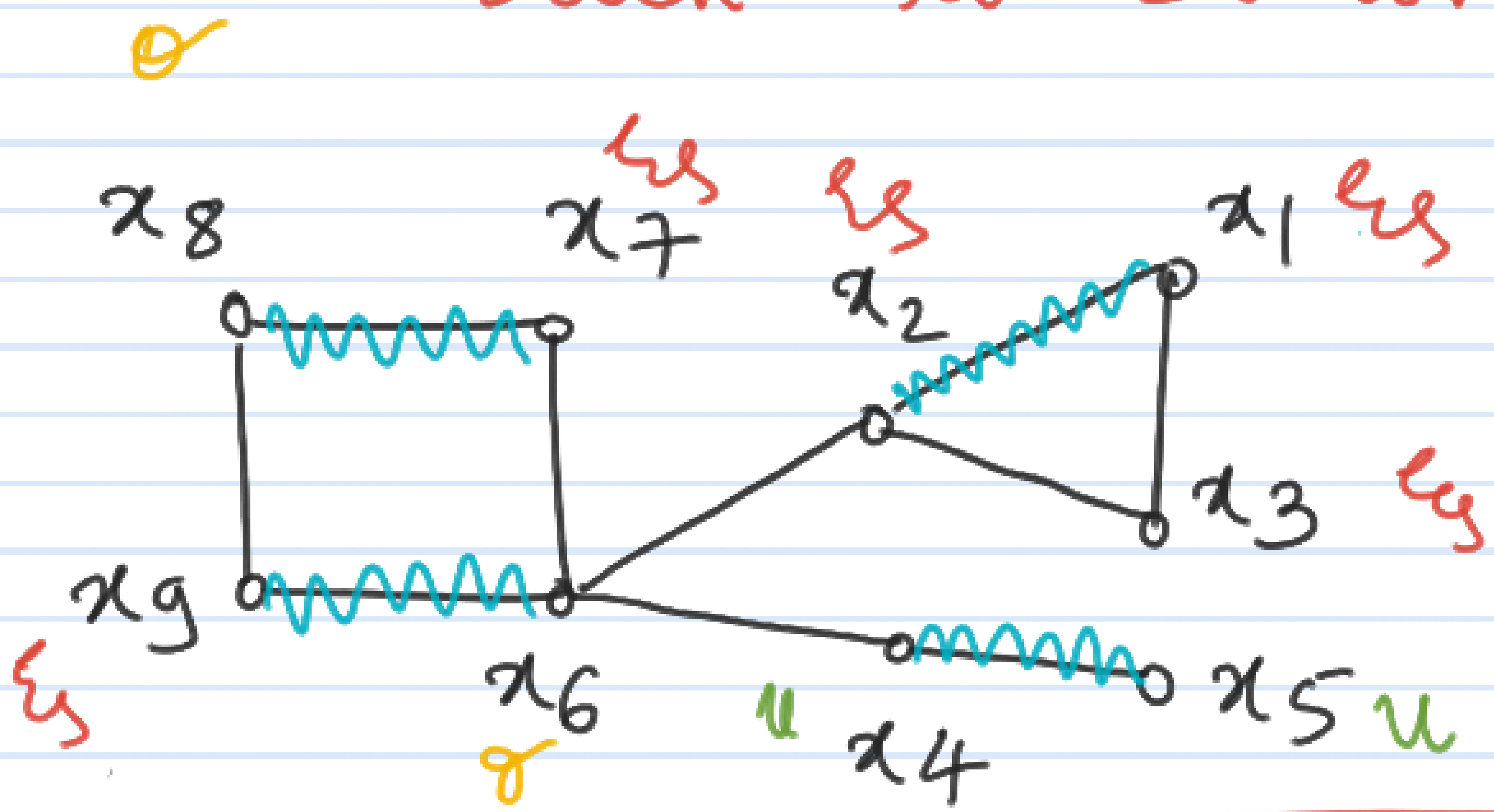
$\mathcal{O}$:

$\mathcal{U}$:

$o(G \setminus S) = 2$ $\qquad$ $o(G \setminus S') = 2$

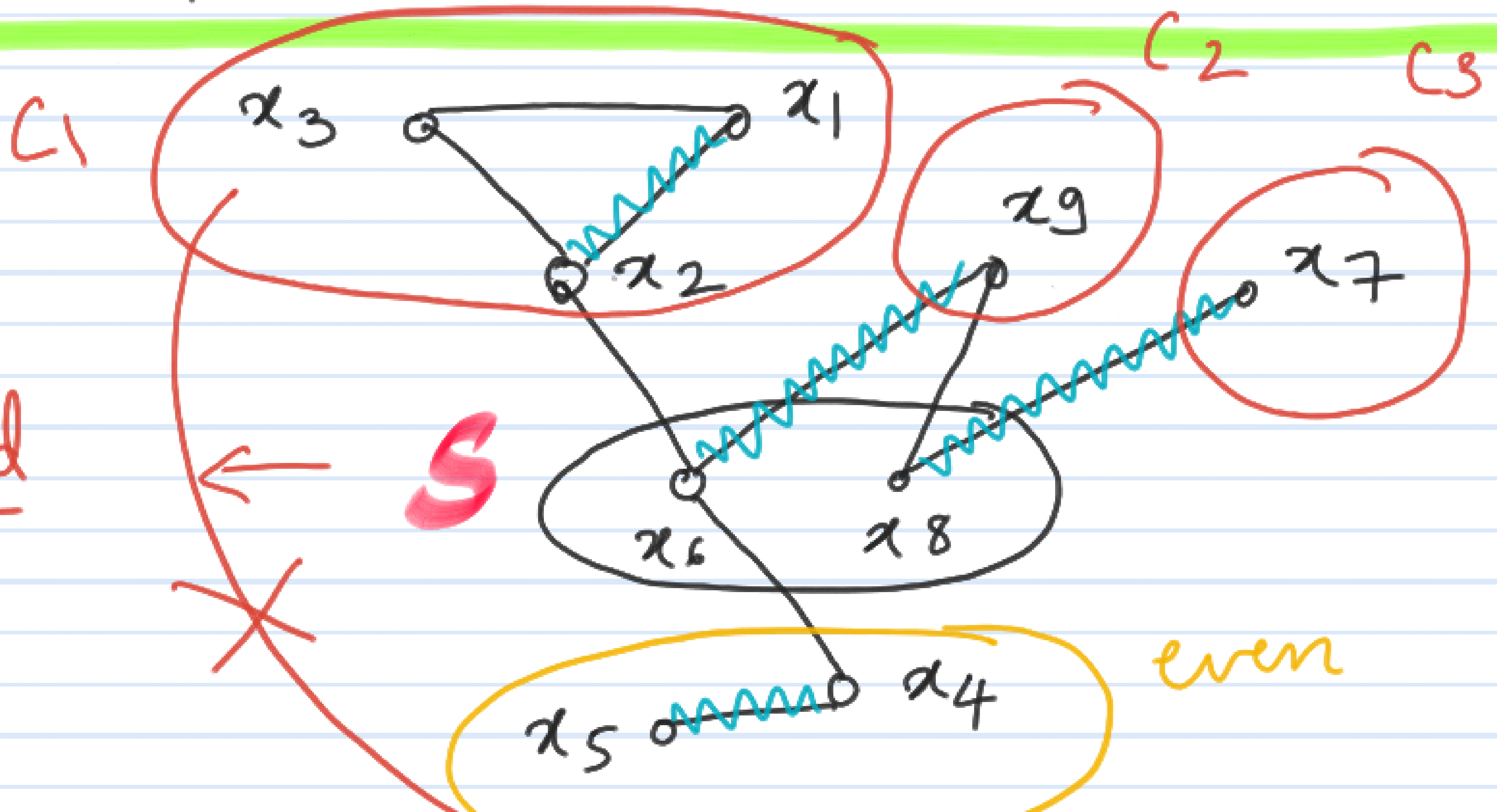
$|S| = 1.$ $\qquad$ $|S'| = 3$

Back to Edmond's Algo : Optimality Certificate



1. Compute max M

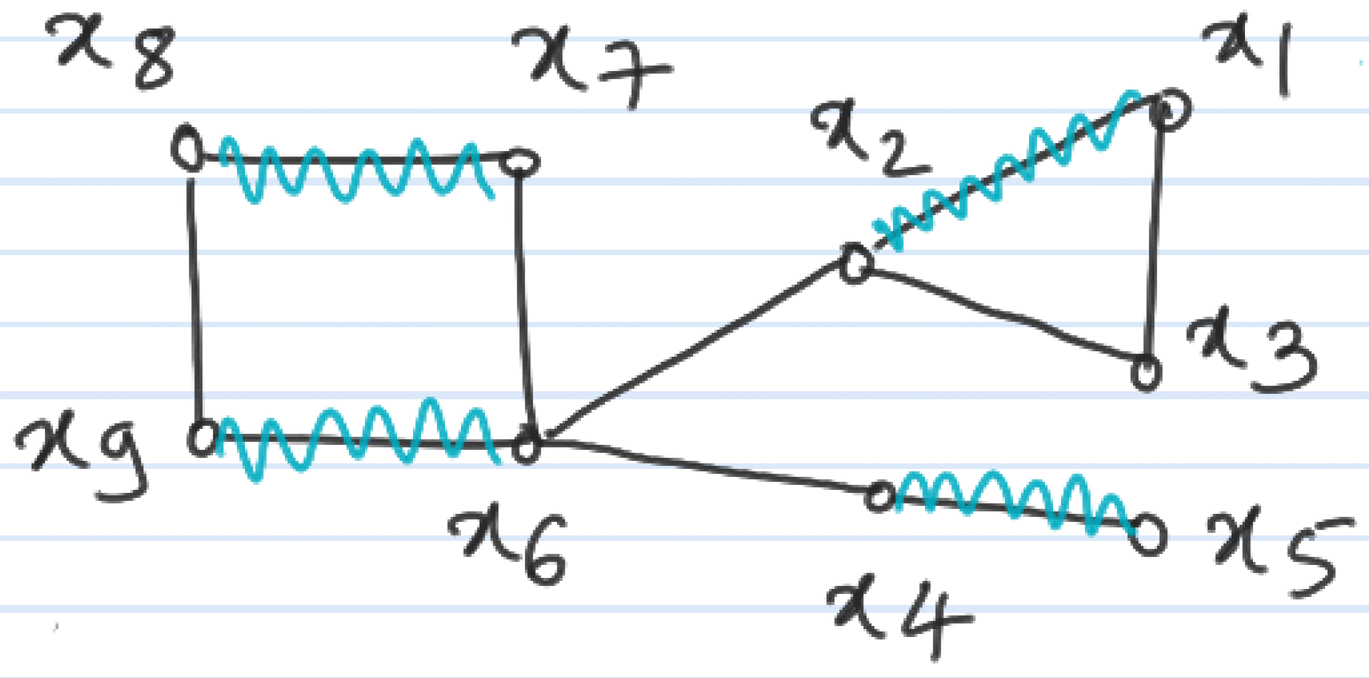
2. Label vertices as $\mathcal{O}_M, \mathcal{E}_M, \mathcal{U}_M$



labelled odd

what is $o(G \setminus S)$?

Back to Edmond's Algo : Optimality certificate



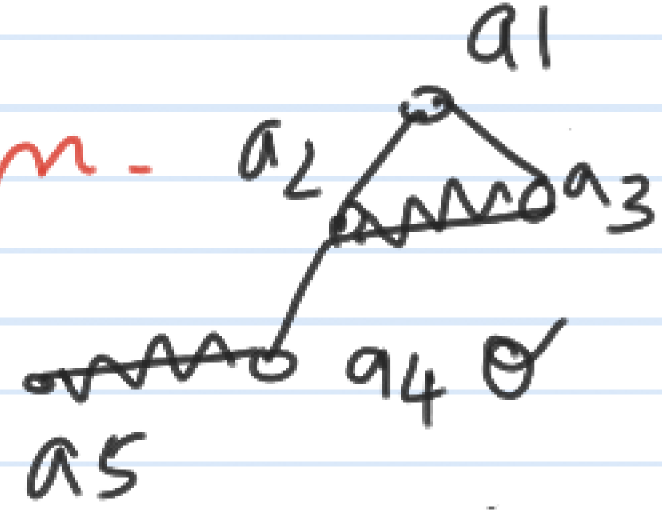
- Run Edmond's algo
- $\mathcal{E}, \mathcal{O}, \mathcal{U}$ be the labels of vertices in the last iteration of algo

Claim : The set $\mathcal{O}$ is witness for Tutte Berge Formula.

that is $\quad |M| = \frac{1}{2}\left[n - \operatorname{def}(\mathcal{O})\right]$

Gallai Edmond's Decomposition Theorem.

eg



M : maximum matching in G.

$\mathcal{E}_M \equiv D(G)$: deficient vertices : some max matching leaves these vertices unmatched

$\mathcal{O}_M \equiv A(G)$: adjacent vertices : neighbours of deficient vertices and not deficient

to do

$\mathcal{U}_M \equiv C(G)$: critical vertices : remaining vertices.

Note : definitions $D(G)$, $A(G)$, $C(G)$ are inv. of max matching , $\mathcal{E}_M$, $\mathcal{O}_M$, $\mathcal{U}_M$ are not.

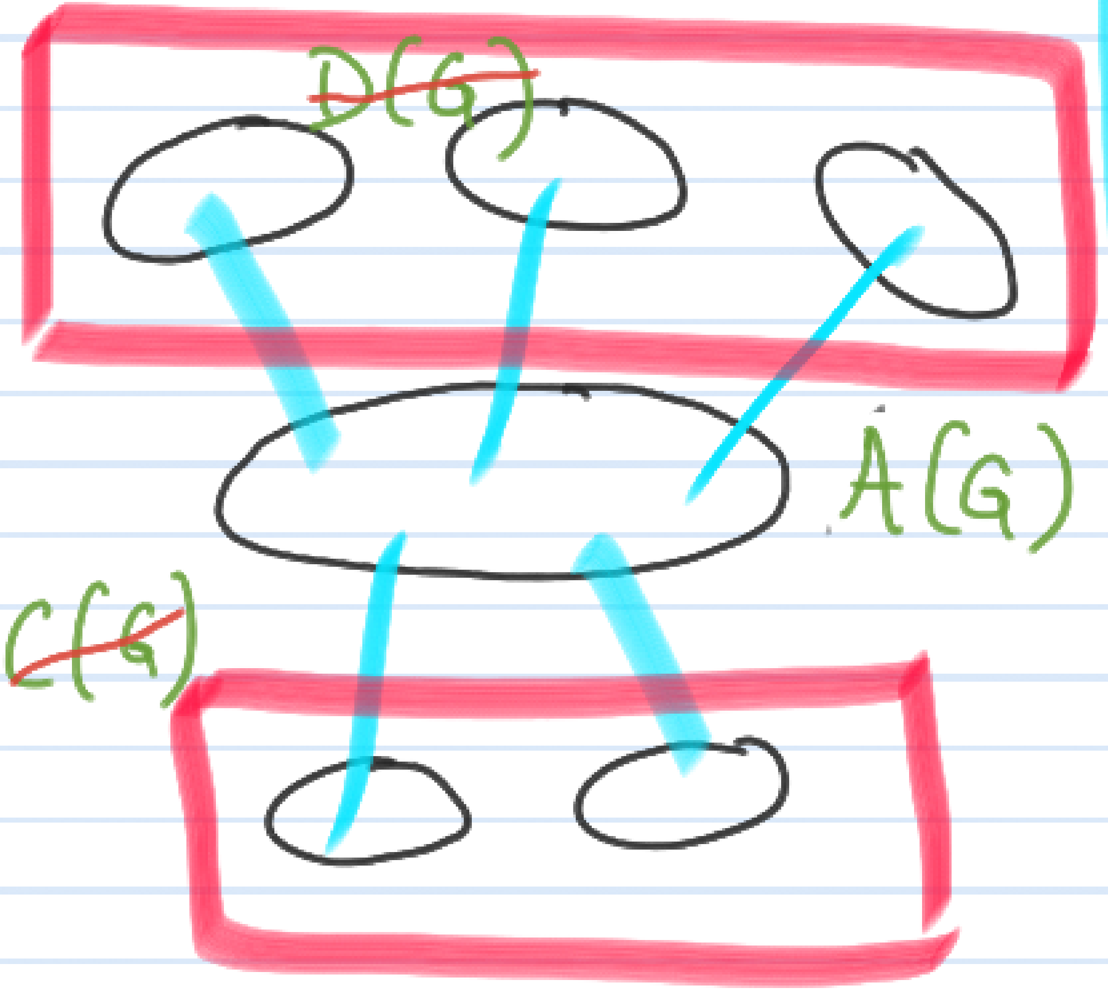
Gallai Edmond's Decomposition Theorem.

M : maximum matching in G.

$\mathcal{E}_G$: $D(G)$

$\mathcal{O}$: $A(G)$

$\mathcal{U}$: $C(G)$



(1) $A(G)$ is witness for Tutte Berge formula

(2) $C(G)$ is made up of even sized components of $G \setminus A(G) = G \setminus \mathcal{O}$

(3) $D(G)$ is made up of odd sized components of $G \setminus A(G)$ and each component is factor critical.

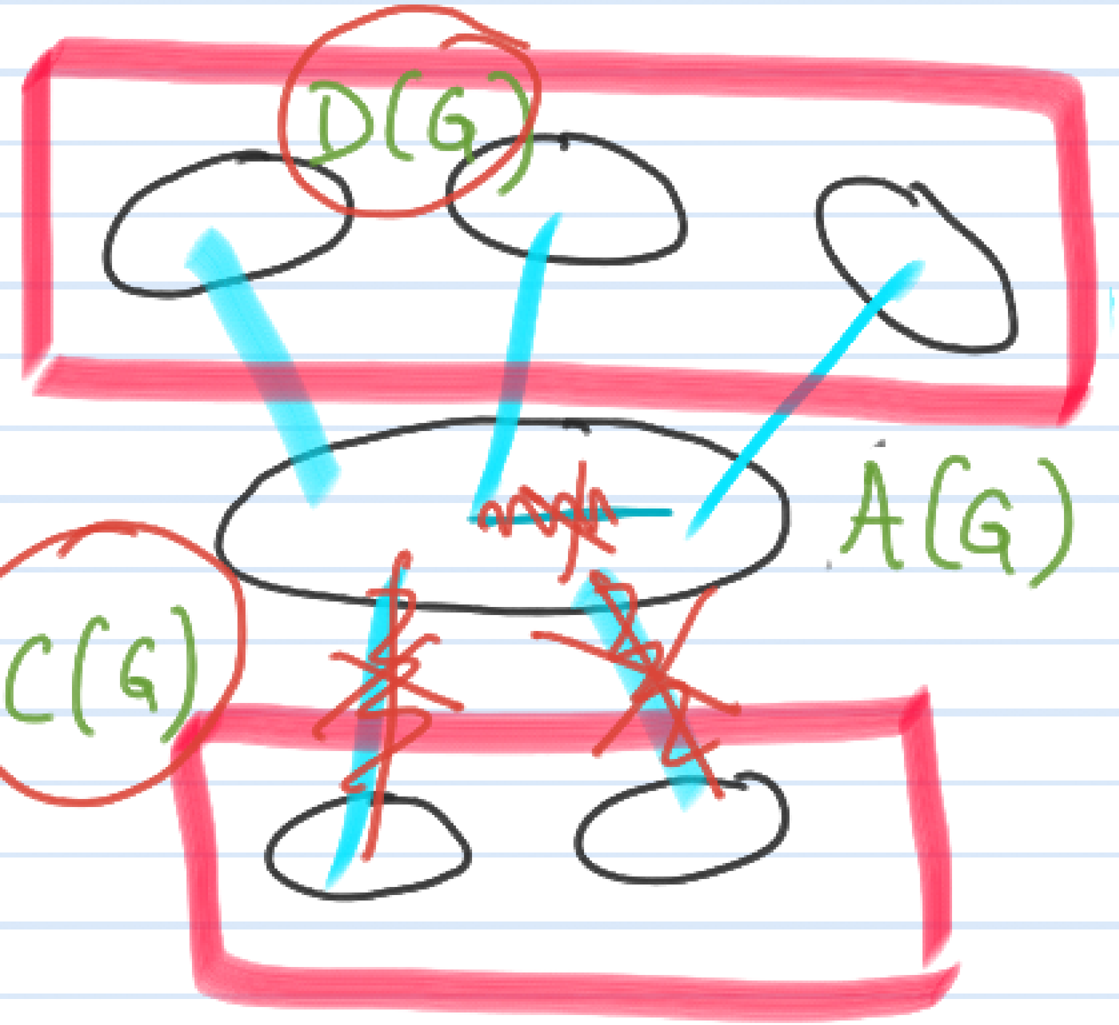
Gallai Edmond's Decomposition Theorem.

M : maximum matching in G.

$\mathcal{E}$: $D(G)$

$\mathcal{O}$: $A(G)$

$\mathcal{U}$: $C(G)$



a — $\mathcal{O}$

$a \to \mathcal{U}$

status in any max matching?

matched

$M(a) = b$

what is label of b?

1) a is odd $\Rightarrow$ b is $\mathcal{E}$

2) a is unreachable $\Rightarrow$ b is $\mathcal{U}$

matched edges are

$\mathcal{E} - \mathcal{O}$

OR

$\mathcal{U} - \mathcal{U}$

OR

$\mathcal{E} - \mathcal{E}$

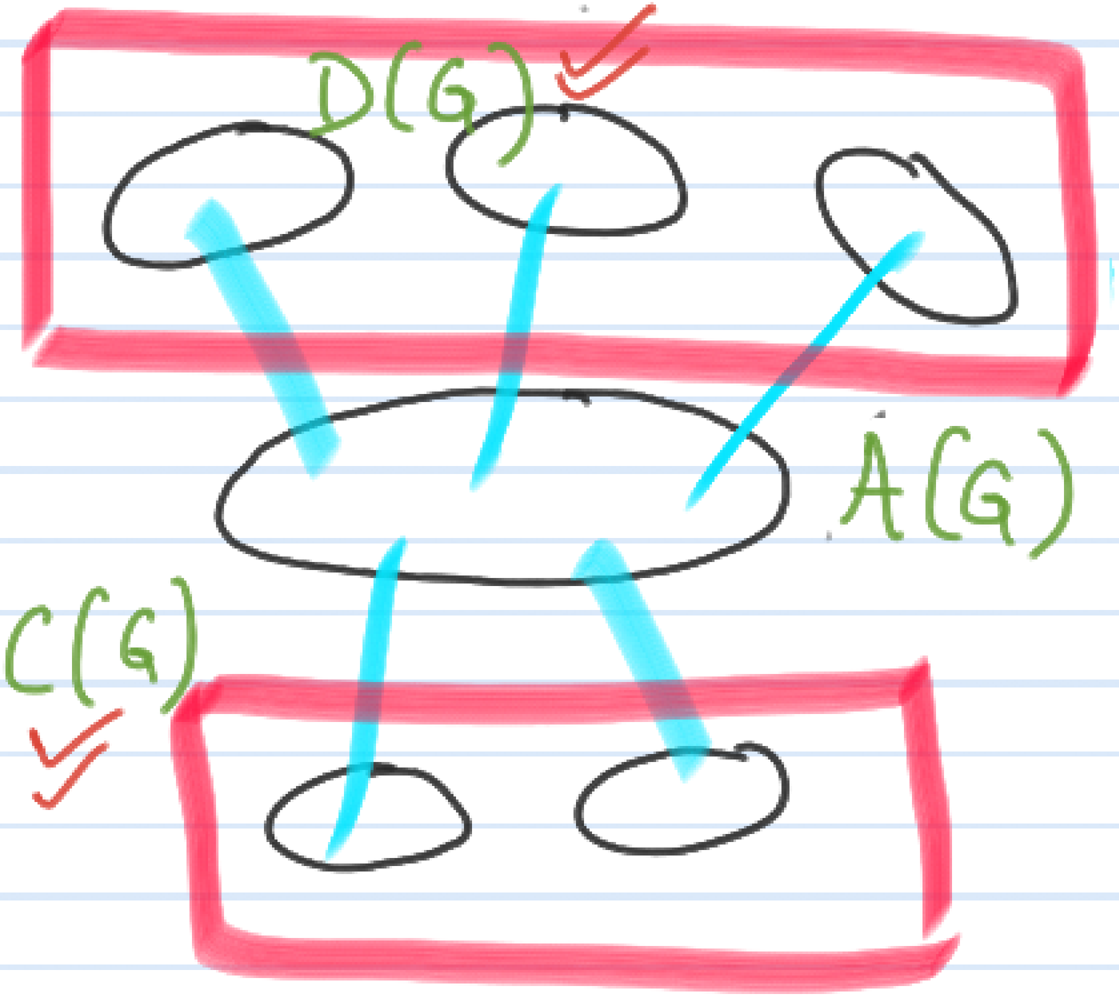
Gallai Edmond's Decomposition Theorem.

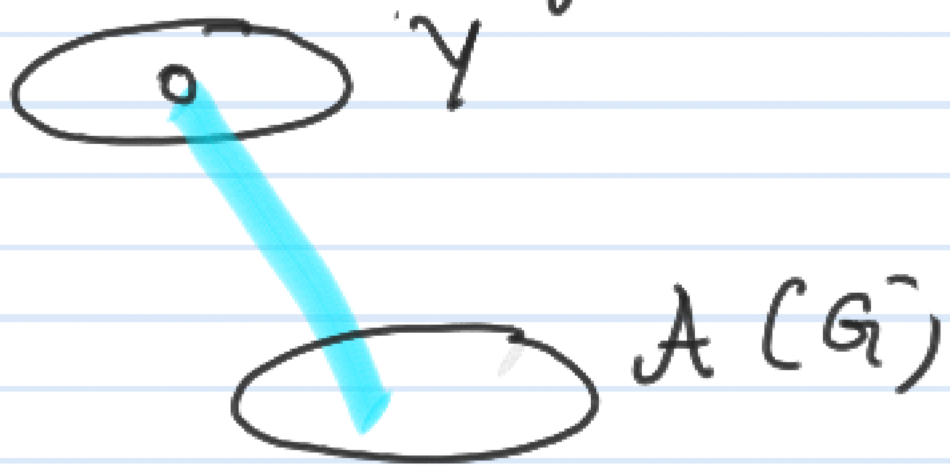
M : maximum matching in G.

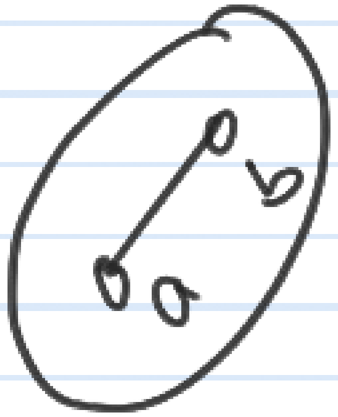
$\mathcal{E}$: D(G)

$\mathcal{O}$: A(G)

$\mathcal{U}$: C(G)







One of the two hold

(i) Y contains exactly one unmatched vertex and blue edges are all unmatched

OR

(ii) Y contains all matched vertices and blue edges contain exactly 1 matched edge.

Factor critical: A graph H is factor critical if for every $v \in V$ $H \setminus v$ admits a perfect matching.

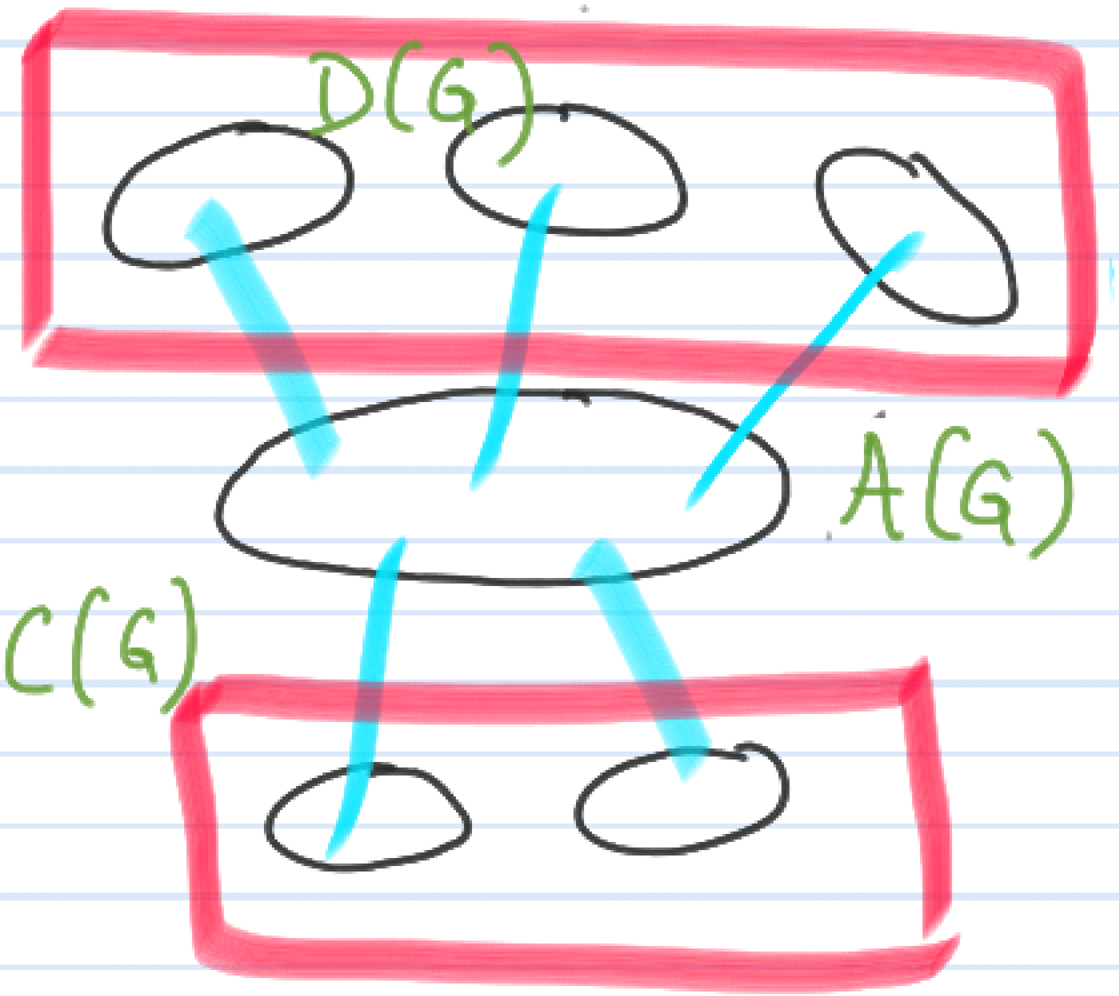
Gallai Edmond's Decomposition Theorem.

M : maximum matching in G.

ε_g : D(G)

θ : A(G)

u : C(G)



size of M

$$|M| \le \frac{1}{2}\left[n - \mathrm{def}(A(G))\right]$$

$$\le \frac{1}{2}\left[n - \left(o(G\setminus A(G)) - |A(G)|\right)\right]$$

$$\le \underbrace{\frac{1}{2}\left[n + |A(G)| - o(G\setminus A(G))\right]}$$

we need to show

$$|M| \geq$$

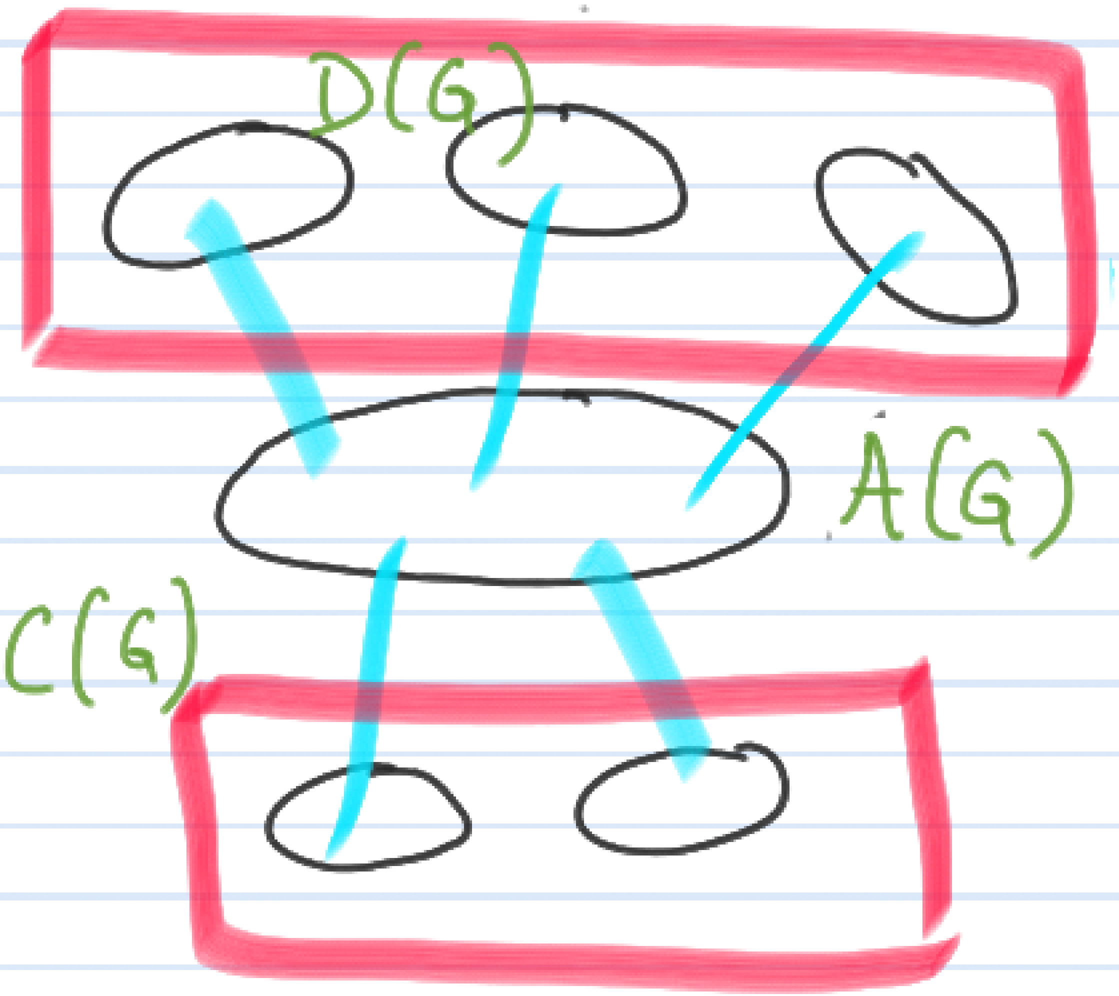
Gallai Edmond's Decomposition Theorem.

M : maximum matching in G.

$\mathcal{E}$: $D(G)$

θ : $A(G)$

$\mathcal{U}$: $C(G)$



size of M $\boxed{\frac{|C(G)|}{2} + \frac{2|A(G)|}{2} + \frac{|D(G)| - o(G \setminus A(G))}{2}}$

$$M = M_{\mathcal{U}} + M_{\mathcal{E}} + M_{\theta}$$

$(\mathcal{U}-\mathcal{U})$ $\quad$ $(\mathcal{E}-\mathcal{E})$ $\quad$ $(\theta-\mathcal{E})$

$C(G)-C(G)$ $\quad$ $D(G)-D(G)$ $\quad$ $A(G)-D(G)$.

$$|M_{\mathcal{U}}| = \frac{|C(G)|}{2} \qquad |M_{\theta}| = |A(G)|$$

$$\frac{|D(G)| - o(G \setminus A(G))}{2}$$