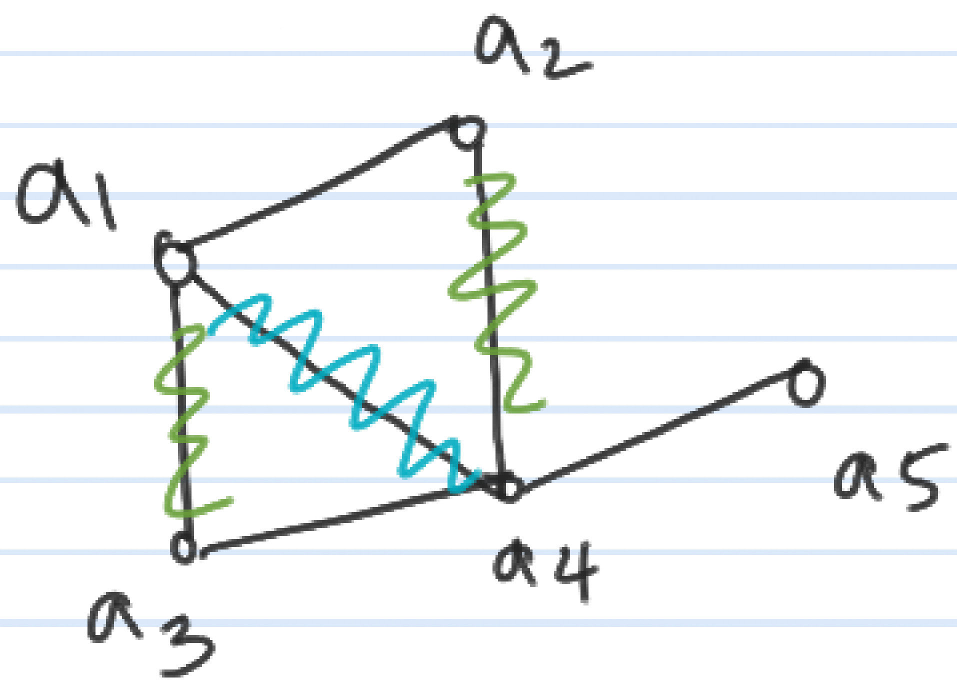
CS 6130 : Advanced Graph Algorithms

- Maximum Matchings
- Bipartite graphs
  - computing a max matching
  - Konigs theorem
  - Hall's theorem
  - A useful decomposition theorem.

Matching in a graph



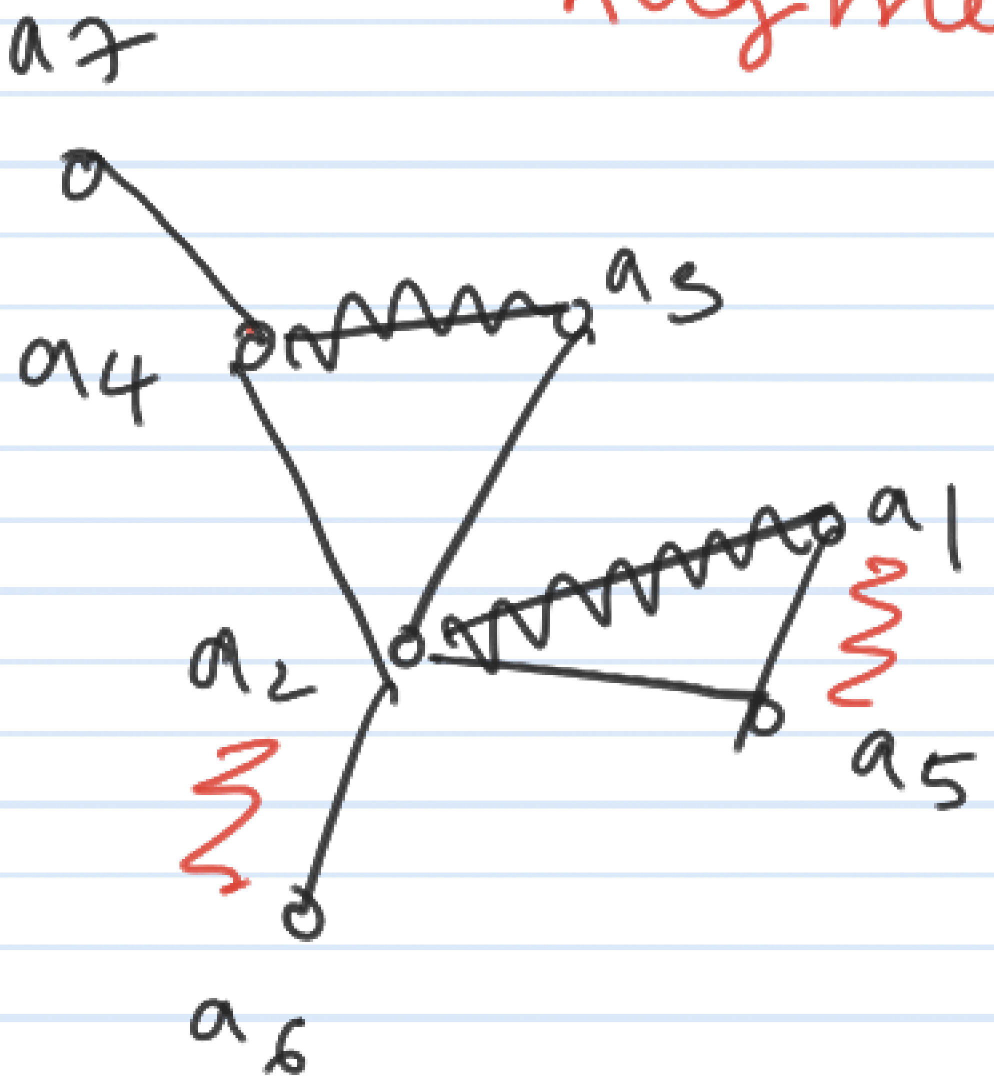
$G = (V, E)$

$M \subseteq E$ s.t. no two edges in M share an endpt

- greedy algo to compute a maximal matching

. Maximal
. maximum

Augmenting paths



Alternating path : simple path with alternate matched and unmatched edges

M

P

$M' = M \oplus P$

Aug. path : Alternating path with both end pts unmatched in M.

Maximum Matching and Berge's thm

- Start with a matching M
- while $\exists$ an aug path P w.r.t M

  $M' = M \oplus P$

  $M = M'$

end while.

Berge's Theorem:

A matching M is maximum iff M has no aug path w.r.t it

Proof of Berge's Theorem

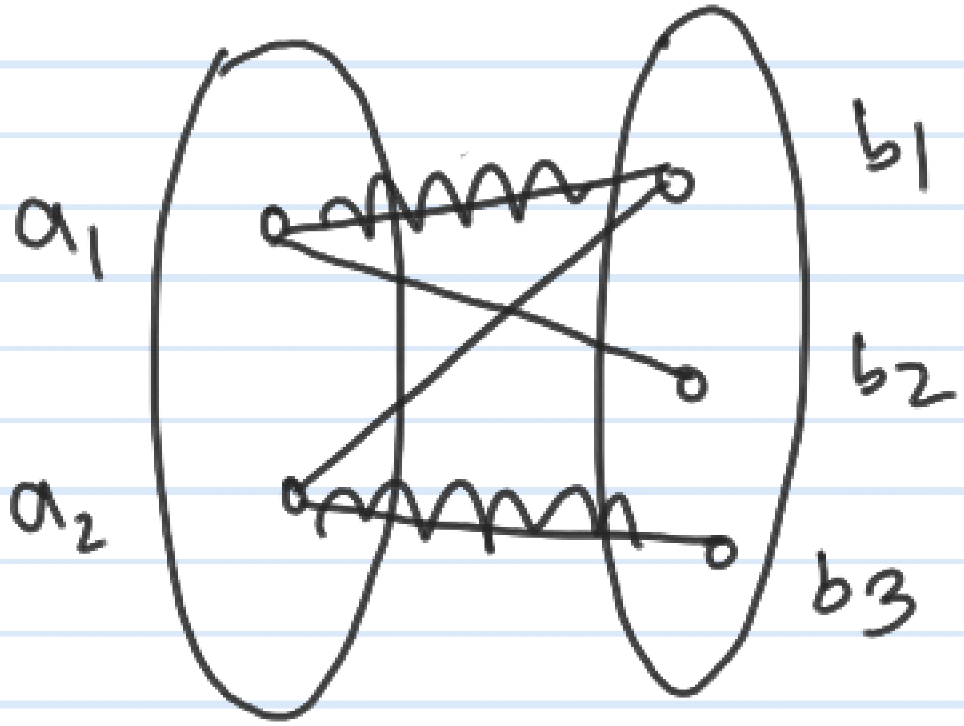
If M does not admit any aug path then M is maximum

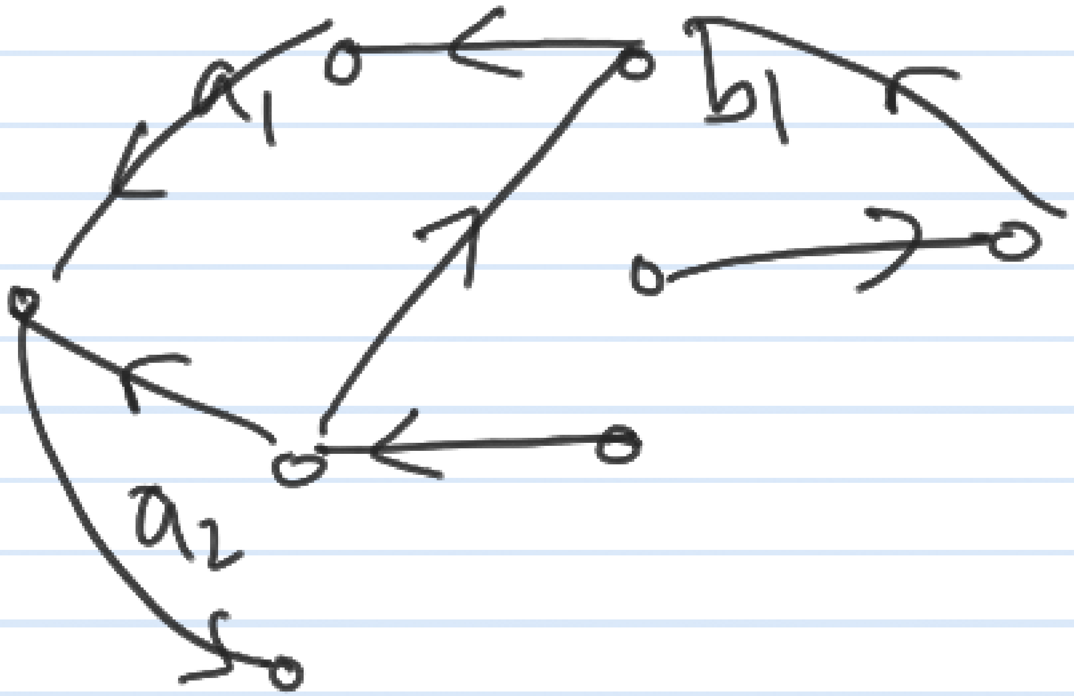
- Assume not, let N be a matching s.t $|N| > |M|$.
- We consider $H = (V, M \oplus N)$
  - ↳ what is the structure of H?
    - ↳ alternating paths and $\underbrace{\text{cycles}}_{}$

all cycles are of even length.

- can all alternating paths be of even length?
  - No, else $|N|$ cannot be $> |M| \Rightarrow$ M aug path exists } * a contradiction

Bipartite Matching and Flows





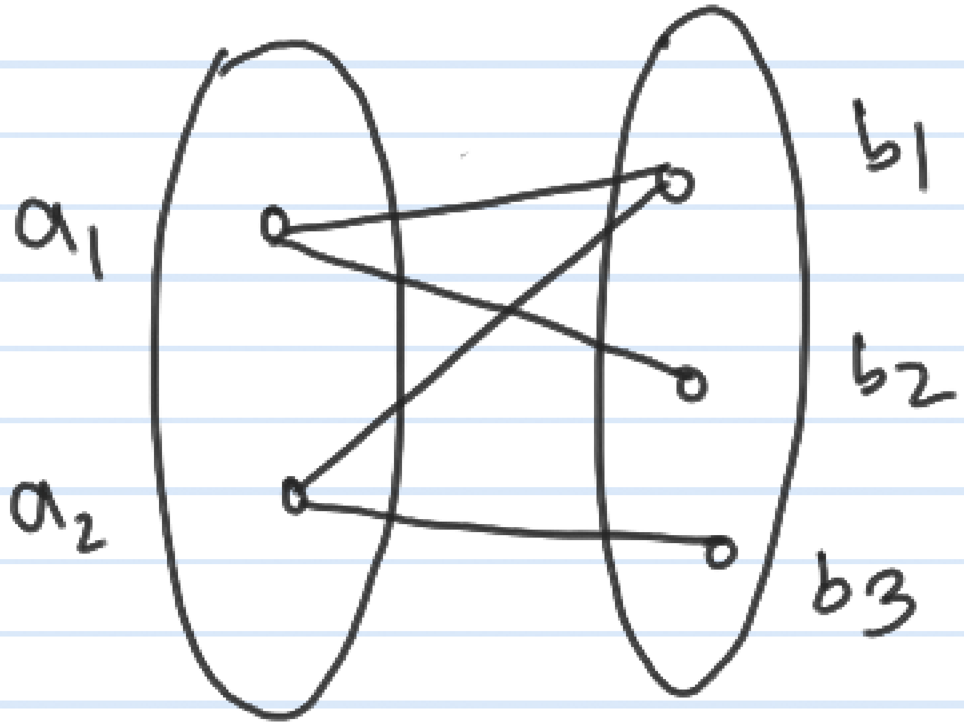
$\underbrace{|f^*|}\cdot O(m+n)$

$|f^*| \leq n \quad O(mn)$

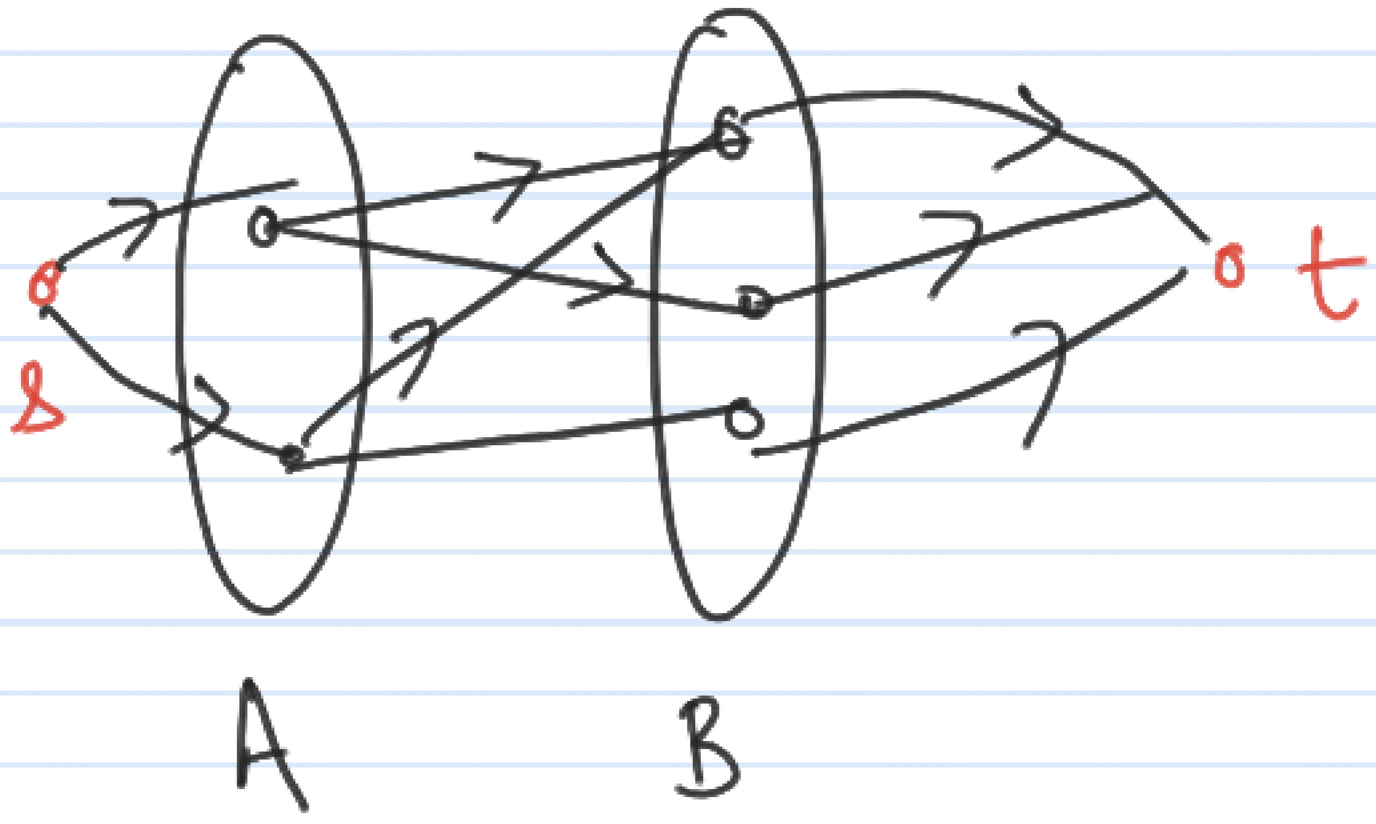
$\underbrace{O(m\sqrt{n})}$

Bipartite Matching and Flows

G

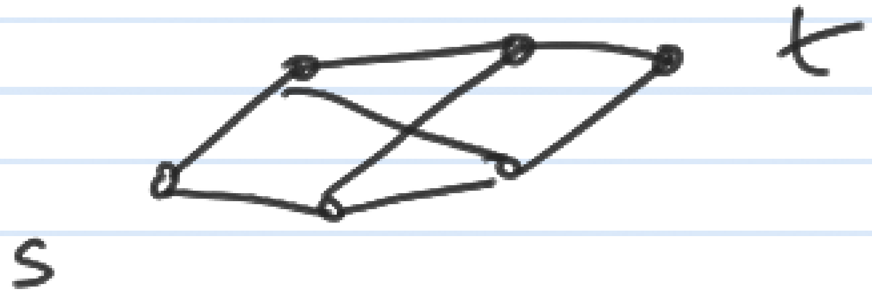


$\mathcal{H}$



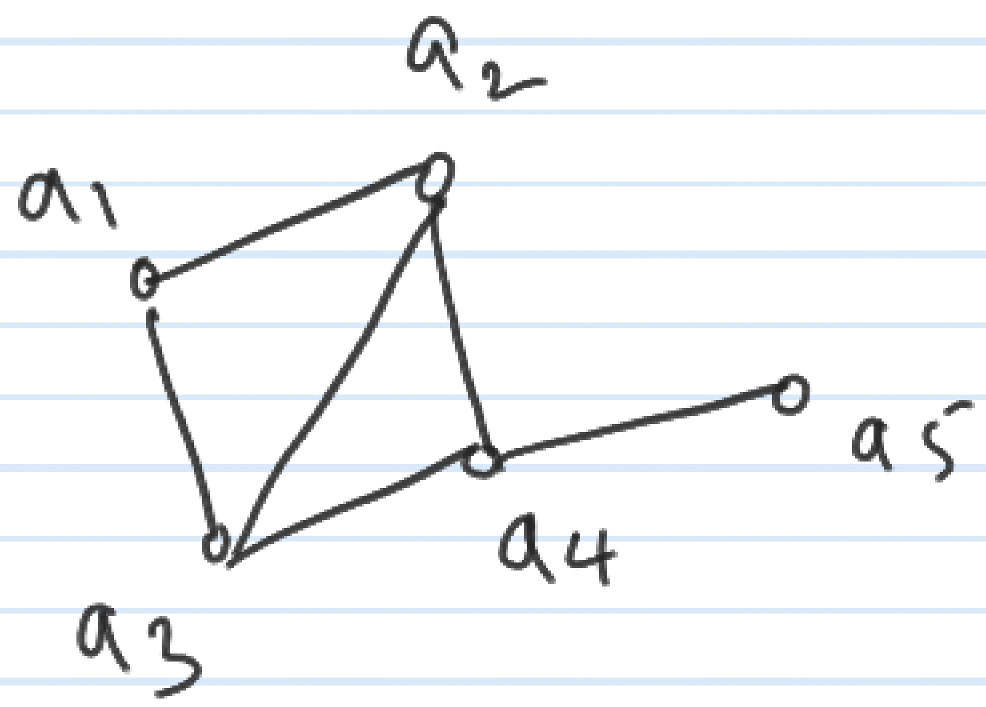
P_1 (unknown solution) $\longrightarrow$ P_2 (known solution)

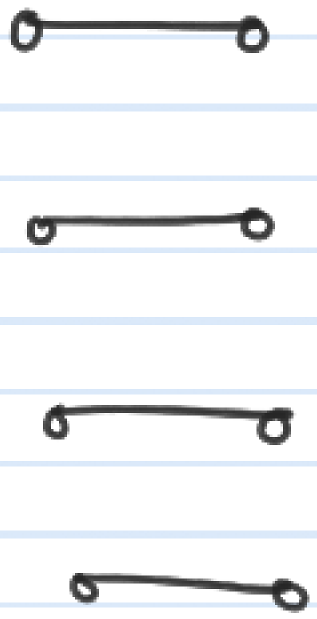
f ;



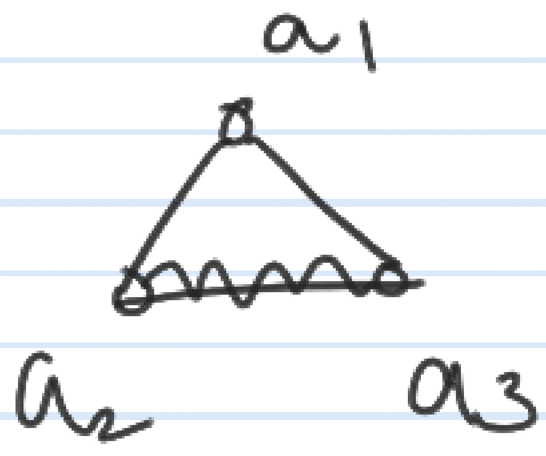
Matching and Vertex Cover

Vertex Cover: A subset of vertices $X \subseteq V$ s.t for every edge $(a, b) \in E$ either a or b or both belong to X.





$|VC| \geq k$



Homework

max matching size $\leq$ min VC

1) Given a maximum matching in a general graph obtain a VC (need not be min VC)

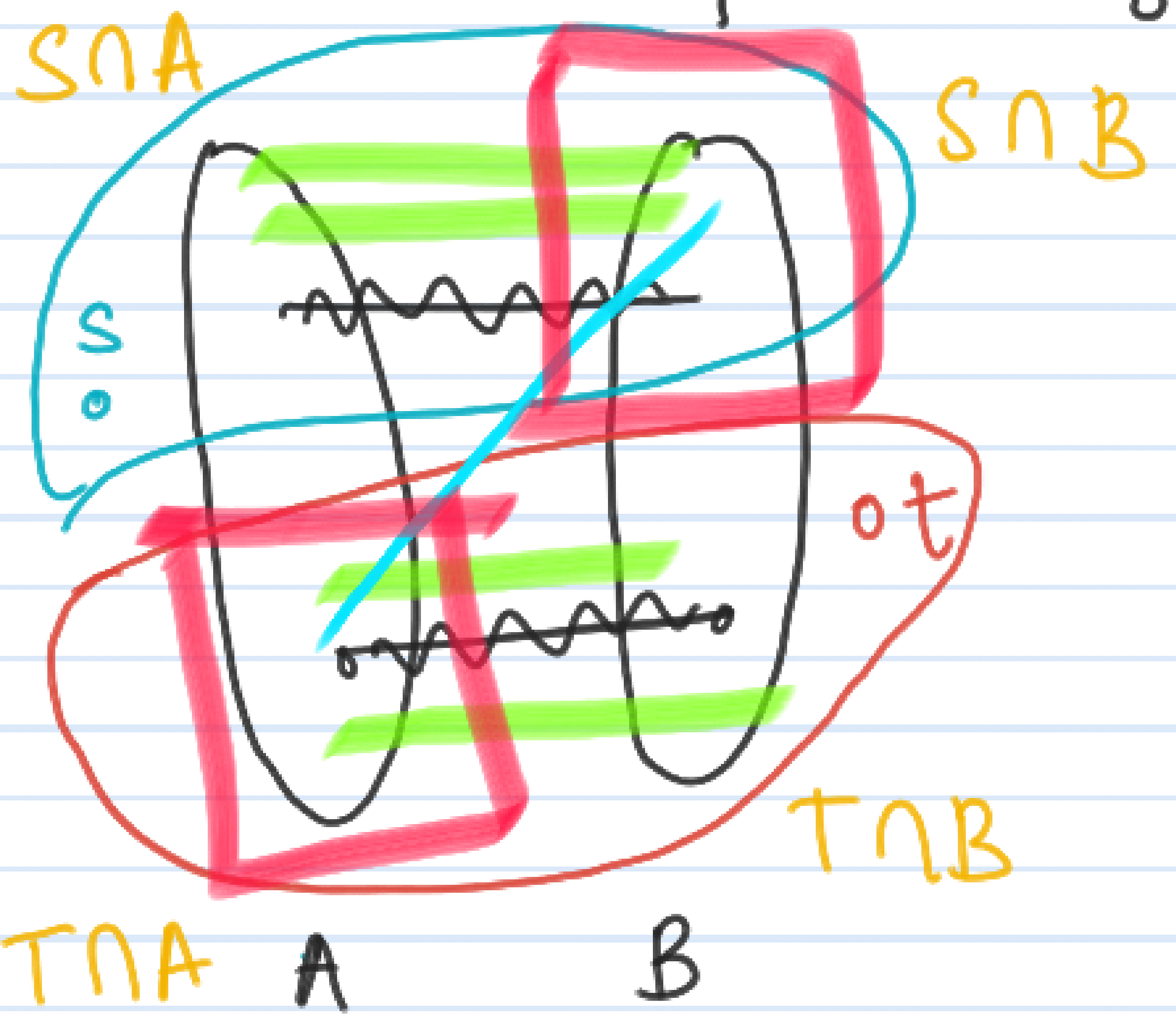
2) We know that G is bipartite iff G does not have any odd cycle

(2a) If G is bipartite then $|M^*| = |VC^*|$

↳ is this an iff statement ?

Proof of Konigs Theorem

For a bipartite graph $|M^*| = |MVC^*|$



$S \cap B$: all vertices are matched

$T \cap A$: " " " "

matched edges : completely in S or
" in T

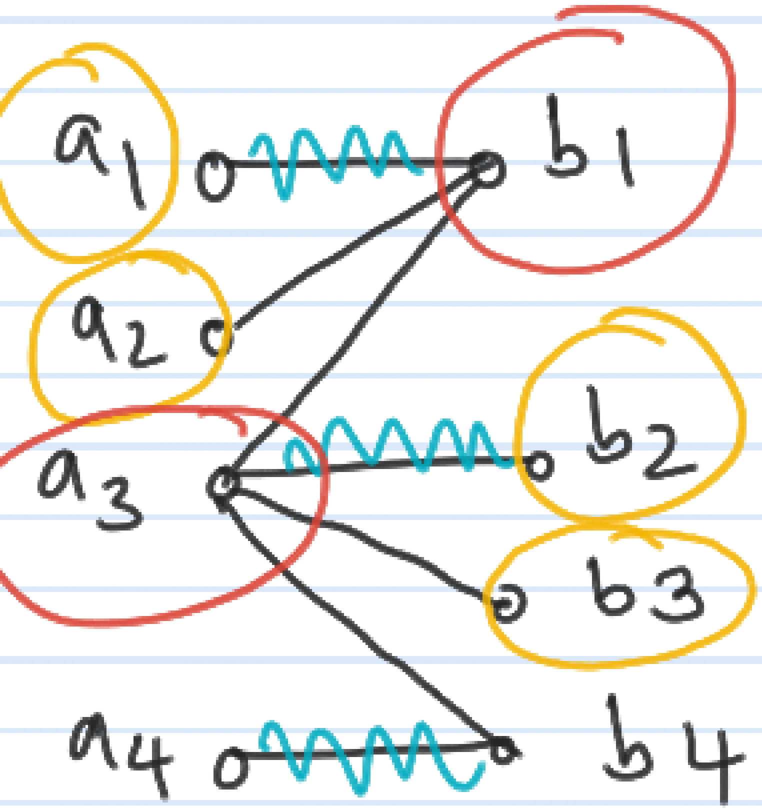
unmatched edges :

$S \cap A \to T \cap B$: no edges

Identify a valid VC

is it min size ? – certificate of optimality

Maximum matchings and their properties



$M_1 = \{ (a_1 b_1) \ (a_3 b_2) \ (a_4 b_4) \}$

$M_2 = \{ (a_2 b_1) \ (a_3 b_3) \ (a_4 b_4) \}$

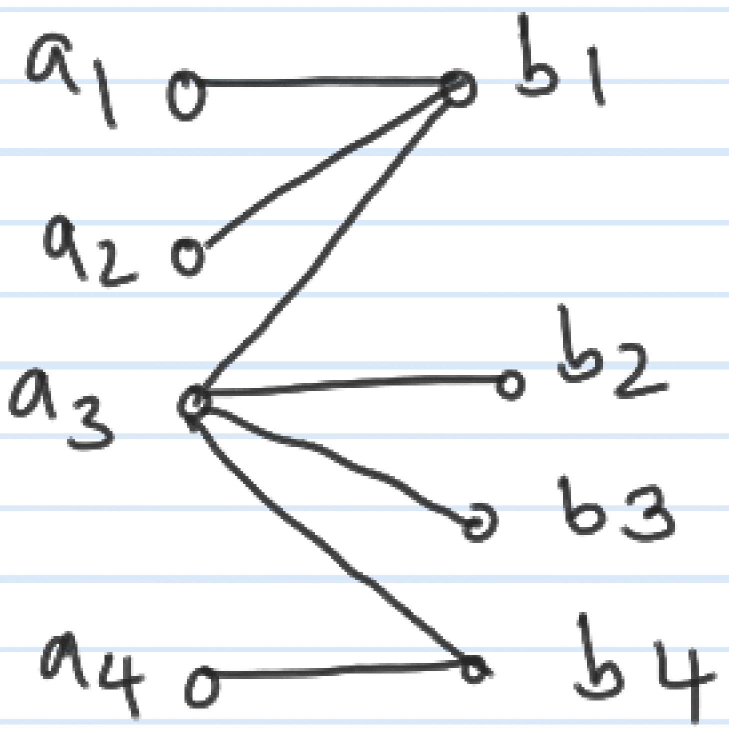
$(a_4 b_4)$ must be matched in <u>any</u> max matching

a_3, b_1, a_4, b_4 must be matched in <u>any</u> max matching

$\mathcal{E}_M$: $u \in A \cup B$ s.t. u is reachable via even length odd

$\mathcal{O}_M$ $\mathcal{U}_M$ alternating path starting at a free vertex in M.

Maximum matchings and their properties



$M_1 = \{(a_1 b_1)\ (a_3 b_2)\ (a_4 b_4)\}$

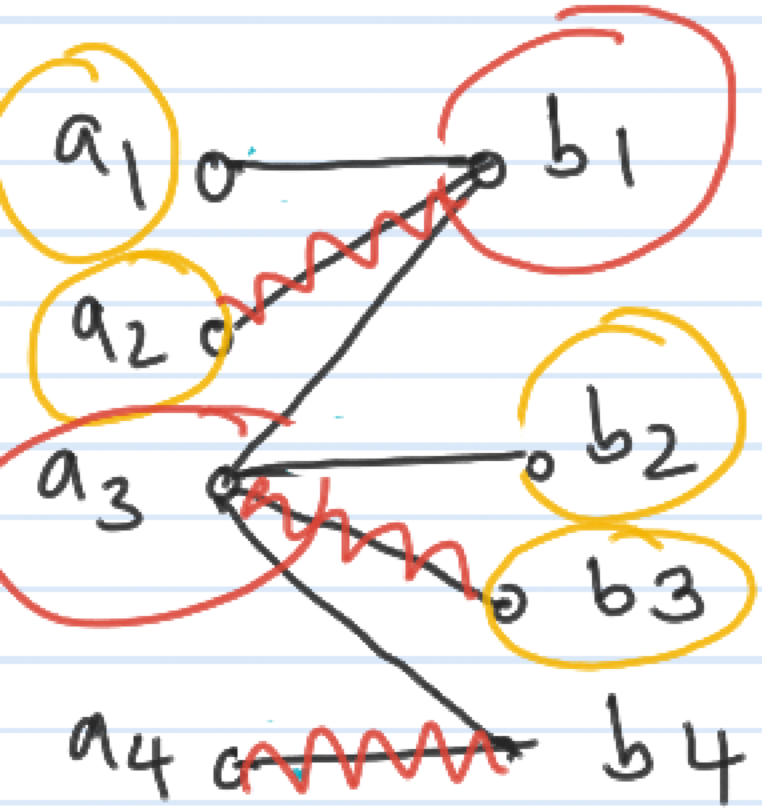
$M_2 = \{(a_2 b_1)\ (a_3 b_3)\ (a_4 b_4)\}$

$\mathcal{E}_M$: even length alternating path starting at a free vertex in M

$\mathcal{O}_M$: odd length alternating path starting at a free vertex in M

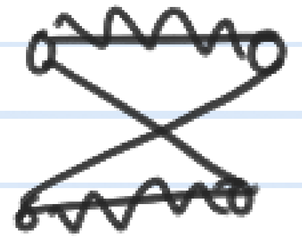
$\mathcal{U}_M$: $(A \cup B) \setminus (\mathcal{E}_M \cup \mathcal{O}_M)$

Maximum matchings and their properties



$M_1 = \{(a_1 b_1)\ (a_3 b_2)\ (a_4 b_4)\}$

$M_2 = \{(a_2 b_1)\ (a_3 b_3)\ (a_4 b_4)\}$



$\mathcal{E}_{M_2} = \mathcal{E}_{M_1}$ $\quad \mathcal{O}_{M_1} = \mathcal{O}_{M_2}$

$\mathcal{E}_{M_1}$: need not be matched in every max mat

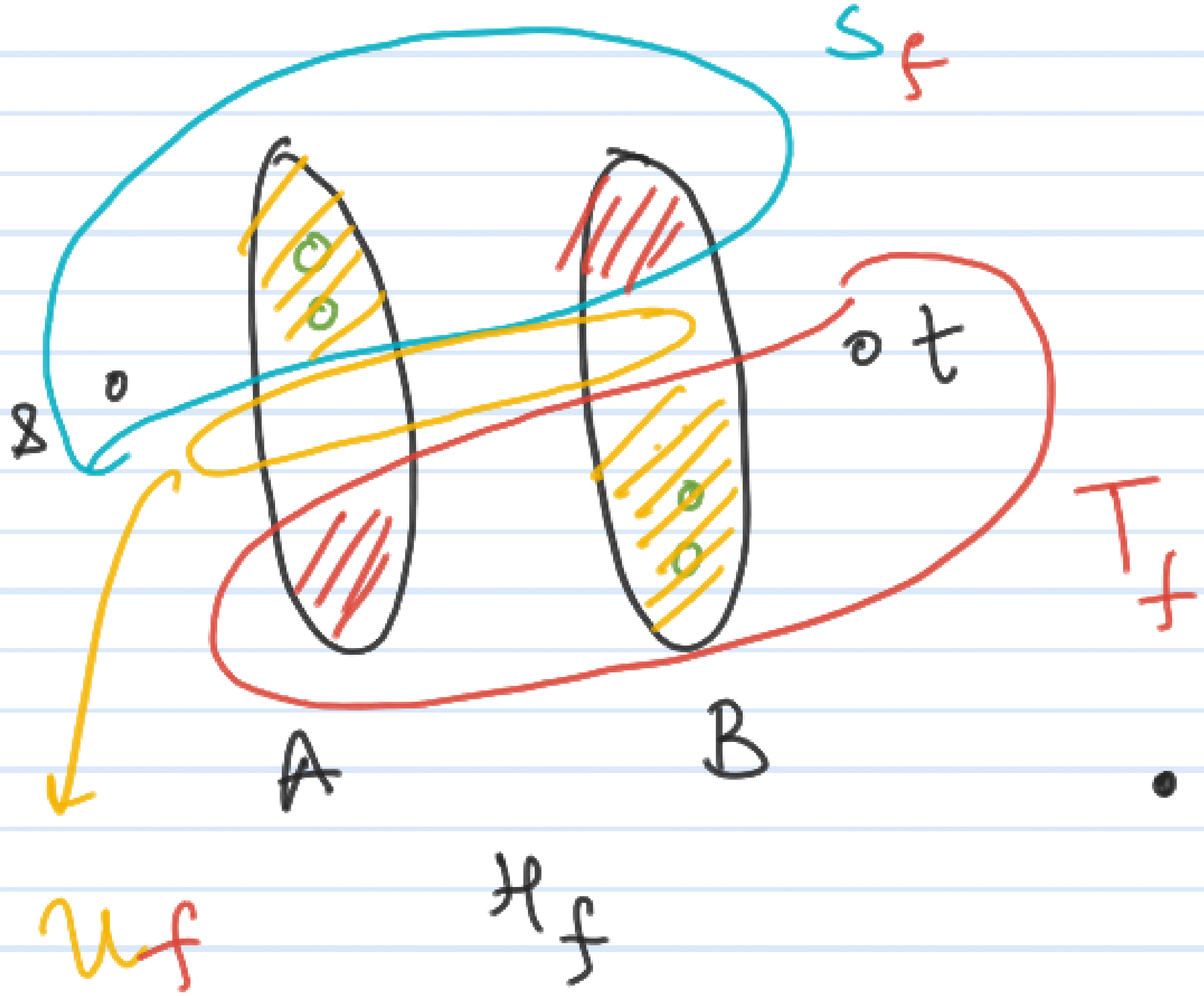
$\mathcal{O}_{M_1}$ $\mathcal{U}_{M_1}$; must be mat. in every max mat

edges between $\mathcal{U}_{M_1} - \mathcal{U}_{M_1}$ must be matched.

Dulmage Mendelsohn Decomposition

- $\mathcal{E}$, $\mathcal{O}$, $\mathcal{U}$ are invariant of max matching
- all vertices in $\mathcal{O}$ an $\mathcal{U}$ are matched in every max matching
- a max matching does not contain any edge of the form $\mathcal{O}\mathcal{O}$, $\mathcal{O}\mathcal{U}$
- G does not contain any edge of the form $\mathcal{E}\mathcal{U}$
- size of max matching is $|\mathcal{O}| + \frac{|\mathcal{U}|}{2}$

Relation between $\mathcal{E}$, $\mathcal{O}$, $\mathcal{U}$ and the corresponding flow n/w.



- Let M be a maximum matching and let f be the corresponding max flow

- S_f, T_f, $\mathcal{U}_f$: Partition of vertices in H_f (residual n/w)

- $\mathcal{E}_M = (S_f \cap A) \cup (T_f \cap B)$
- $\mathcal{O}_M = (S_f \cap B) \cup (T_f \cap A)$

$\mathcal{U}_M = \mathcal{U}_f$

prove these.

Decomposition of vertices w.r.t a max flow.

- H be any flow n/w

  f_1 and f_2 are 2 max flows in H

  $S_1\ T_1\ U_1$: partition w.r.t f_1

  $S_2\ T_2\ U_2$: partition w.r.t f_2

Claim: $S_1 = S_2$, $T_1 = T_2$, $U_1 = U_2$

Decomposition of vertices w.r.t a max flow

Proof: $S_1 = S_2$ 　　　　Two parts $S_1 \subseteq S_2$

$S_2 \subseteq S_1$

Useful Facts :

$(S, T \cup U)$ is a min cut

For $\boxed{\text{any}}$ max flow f in H and $\boxed{\text{any}}$ min(X, Y) cut

(1) $a \circ\!\!\longrightarrow\!\!\circ\, b$ 　 $f(a,b) = c(a,b)$ 　 forward edge

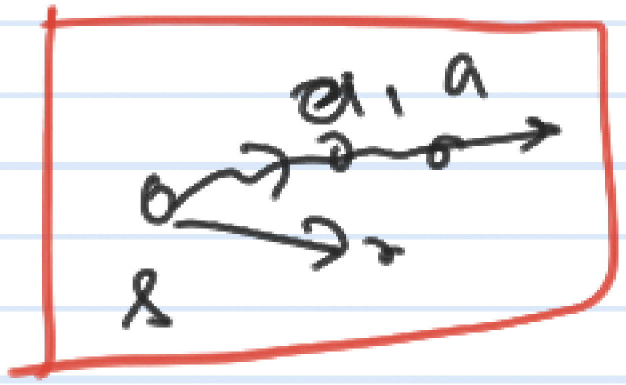
(2) $b \circ\!\!\longleftarrow\!\!\circ\, a$ 　 $f(a,b) = 0$ 　 reverse edge

Decomposition of vertices w.r.t a max flow

Proof: $S_1 = S_2$

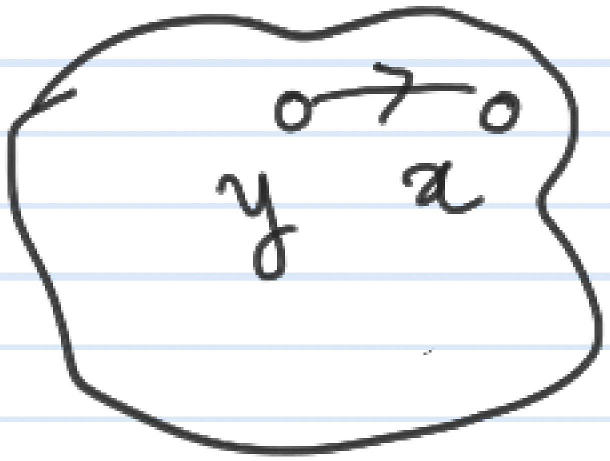
Two parts $S_1 \subseteq S_2$

$S_2 \subseteq S_1$



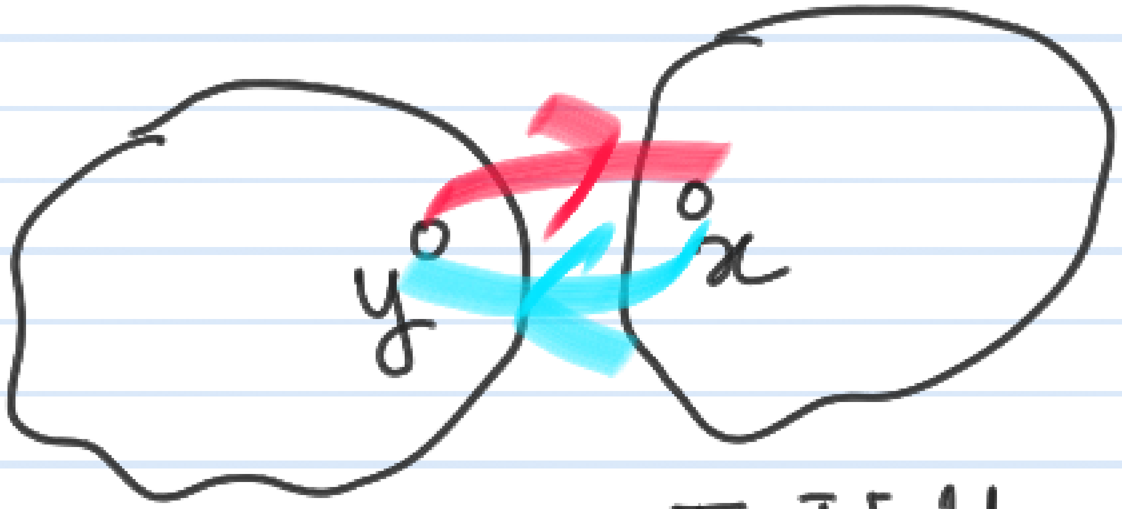
Goal: $x \in S_1 \implies x \in S_2$

Assume not : $\therefore$ $x \in T_2 \cup U_2$



S_1

H_{f_1}



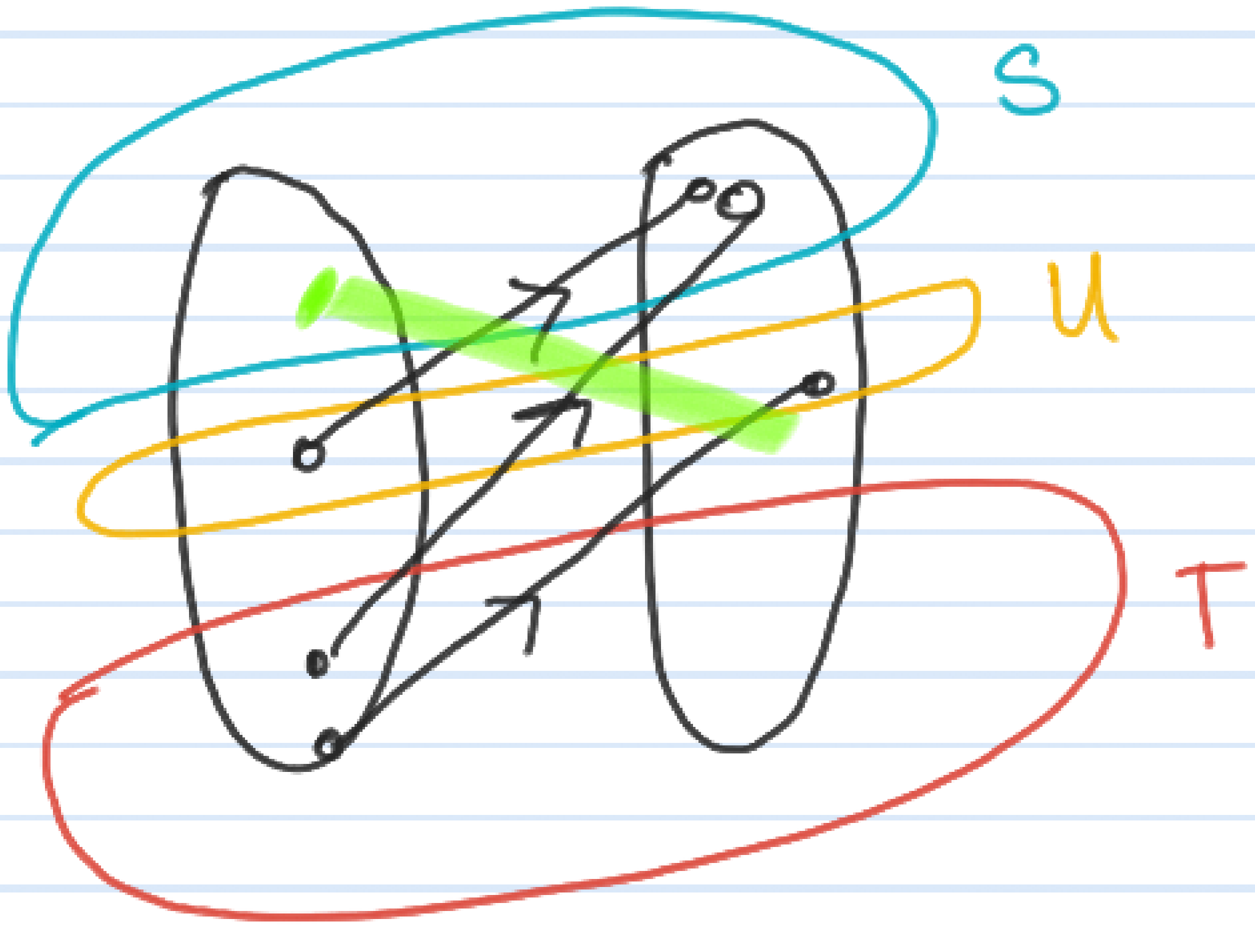
S_2 $T_2 \cup U_2$

Why is $y \to x$ present in H_{f_1}?

(1) $y \to x$ is in H

(2) $x \to y$ is in H

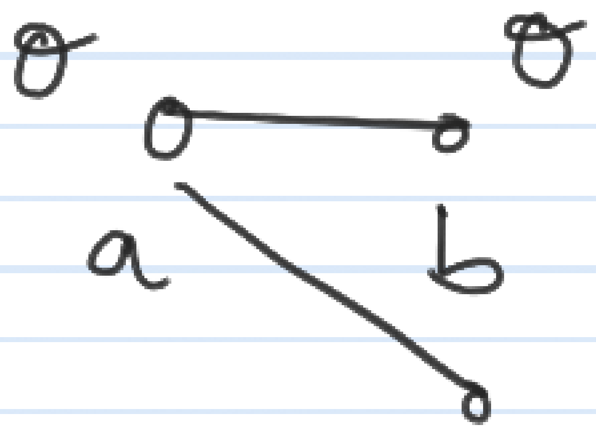
Back to Dulmage Mendelsohn Decomposition



$$\mathcal{E}_M = (S\cap A) \cup (T\cap B)$$

$$\mathcal{O}_M = (S\cap B) \cup (T\cap A)$$

$$\mathcal{U}_M = \mathcal{U}$$



1) Invariance w.r.t. matching ✓
2) no max matching contains an $\mathcal{O}\mathcal{O}$ or $\mathcal{O}\mathcal{U}$ edge ✓
3) G does not contain any $\mathcal{E}\mathcal{U}$ edge
4) size of matching $|\mathcal{O}| + |\mathcal{U}|/2$

Flows and Bipartite matchings

- Very closely related
- Konigs theorem, Halls theorem, DM decomposition
  ↳ easy corollaries of max flow min cut theorem
- Allows generalizations of these theorems for capacitated matching problems.