Popular matchings in one-sided pref. setting

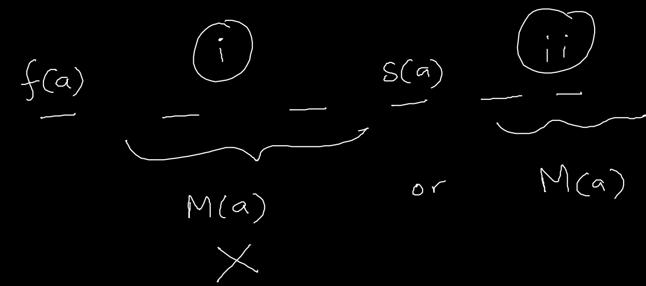
Recap [strict] characterization of pop. matchings:

G: instance. M is a pop. matching in G iff

1. every f-post q is matched in M s.t. $M(q) \in f(q)$
2. $\forall a \in A$, $M(a) \in \{f(a), s(a)\}$

Necessity proof for 2

Pf: Sps. 2 doesn't hold.



(i) $M(a)$ is s.t. $f(a) \succ_a M(a) \succ_a s(a)$

Claim: $q = M(a)$ is an f-post

By 1 $\Rightarrow M(q) \in f(q) \Rightarrow a \in f(q) \Rightarrow f(a) = q \Rightarrow \perp$ because $f(a) \succ_a q = M(a)$

(ii) $s(a) \succ_a M(a)$, $q = s(a)$

Goal: construct N : $N \succ M$

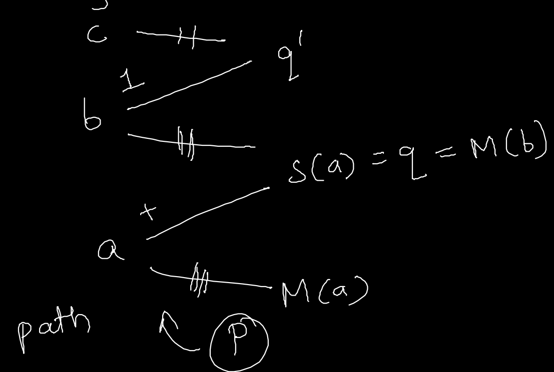
if q is unmatched : $N = M \setminus \{(a, M(a))\} \cup \{(a, q)\}$

$\Rightarrow N \succ M \Rightarrow \perp$

if q is matched : $b = M(q)$

Q: Is $q = f(b)$: No

$\exists q'$ s.t. $q' = f(b)$



if q' is unmatched : $N = M \oplus P$

else : $c = M(q')$: free c, $M \oplus P$

$\Big\}$ $N \succ M \Rightarrow \perp$

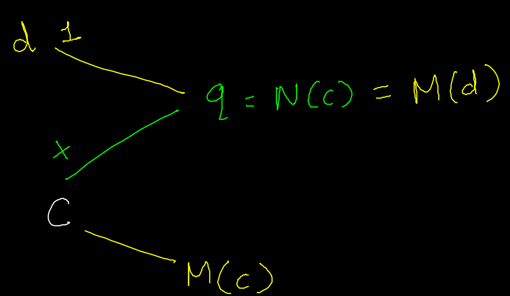
Sufficiency: To show: If M satisfies ① & ② then M is a pop. matching

Pf: Sps. M is not a pop. matching

$\Rightarrow \exists\, N$ s.t. $N \succ M$

$\Rightarrow \exists$ app. c s.t. $N(c) \succ_c M(c)$

Goal: To show a distinct app. d s.t. $M(d) \succ_d N(d)$ for every $c : N(c) \succ_c M(c)$



$M(c) \in \{ f(c), s(c) \}$ (by ②)

because $q \succ_c M(c) \Rightarrow M(c) \neq f(c)$

$\Rightarrow M(c) = s(c)$

$\Rightarrow q = f\text{-post} \Rightarrow M(q) \in f(q)$ (by ①)

let $d = M(q) \in f(q) \Rightarrow f(d) = q$

$\Rightarrow d : M(d) \succ_d N(d)$

$d = M(N(c)) \Rightarrow$ distinct

$a_1 : p_1 \;\; p_2 \;\; q_1$

$a_2 : p_1 \;\; p_2 \;\; q_2$

$a_3 : p_1 \;\; p_2 \;\; q_3$

$a_4 : p_1 \;\; p_2 \;\; q_4$

$M = \{ (a_1, p_1), (a_2, p_2), \overline{(a_3, q_3)}, \overline{(a_4, q_4)} \}$

Q: Is M a pop. matching?

No

$N = \{ (a_1, p_1), (a_2, p_2) \}$

$M \succ N$ Is M a pop.

$|f\text{-posts} \cup s\text{-posts}| = 2$

$|A| > 2$

$a_1 : p_1 \;\; \ell_1$ $\quad M_i = (a_i, p_1)$

$a_2 : p_1 \;\; \ell_2$

$a_3 : p_1 \;\; \ell_3$

$a_4 : p_1 \;\; \ell_4$ Q: Does it admit a pop. matching

Ties in the lists

$\swarrow G$

a_1:	$(p_1\ p_2)$	p_4	ℓ_1	
a_2:	p_1	$(p_2\ p_5)$	ℓ_2	
a_3:	p_2	$(p_4\ p_6)$	ℓ_3	
a_4:	p_2	p_1	p_3	ℓ_4
a_5:	p_4	p_3	p_2	ℓ_5
a_6:	$(p_5\ p_6)$	p_1	ℓ_6	

M is a pop. matching.

$\underline{Q}$: Is every f-post q is matched s.t. $M(q) \in f(q)$

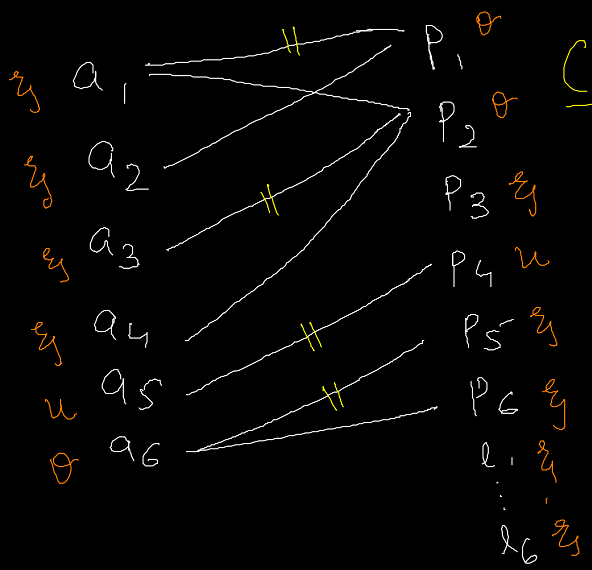
$\underline{A}$: Need not be true [ties]

a_6: $f(a_6) = s(a_6)$

f-posts $\cap$ s-posts $= \emptyset$ [strict]

$E(G_1)$: E_1

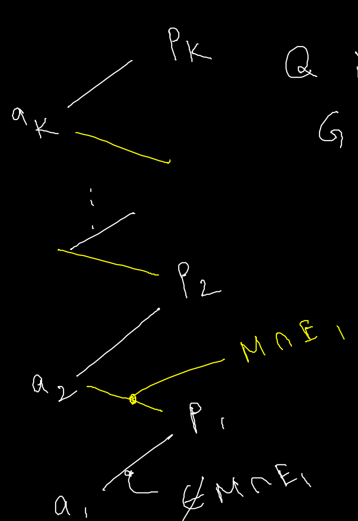
First-choice graph (G_1): $(a,p) \in G_1$ iff $p \in f(a)$



Claim (A) : M is a pop. matching in G. Then $M \cap E_1$ is a maximum matching in G_1.

Pf of (A) : Sps not. $M \cap E_1$ is NOT a maximum matching in G_1.

$\Rightarrow \exists$ an augmenting path Q w.r.t. $M \cap E_1$ in G_1



Q in G_1

(i) $(a,p) \in Q \Rightarrow (a,p) \in E_1$

(ii) $p_1 \ \dots \ p_K$: which of these are f-posts?

All of them

(iii) $M(a_1) \in f(a_1)$? No

Goal : construct N s.t. $N \succ M \Rightarrow \perp$

Redefine $s(a)$ for "ties":

Recall [strict] : $s(a)$: top-pref non-f-post for a

Goal: $\forall a$, $M(a) \in f(a) \cup s(a)$

Ties : $s(a)$ = top-pref "even" post for a in G_1

Characterization in ties [pop. matchings] (Abraham et al. '2007)

M is a pop matching in G iff

1. $M \cap E_1$ is a maximum matching G_1
2. $M(a) \in f(a) \cup s(a)$ [$s(a)$: redefined]