Popular matchings in one-sided preference setting

Recap : Popularity in the SM setting (majority : >)

$M \succ N$ i.e. M is more popular than N if # votes that M gets > # votes that N gets

M is a popular matching if there is no other matching N s.t. $N \succ M$

One-sided preference A : applicants , P : posts

Only applicants have preference ordering over posts

votes $\equiv$ # votes by applicants

$M \succ N$ i.e. M is more popular than N if # votes by applicants that M gets > # votes by applicants that N gets

M is a popular matching if there is no other matching N s.t. $N \succ M$

$a_1 : p_1 \; p_2$

$a_2 : p_1$

$M = \{a_1 p_1\}$

$N = \{a_1 p_2, \; a_2 p_1\}$

	M	N
a_1	✓	
a_2		✓

$M \succ N$? ✗

$N \succ M$? ✗

M & N : Both are popular matchings

$a_1 : p_1 \; p_2$

$a_2 : p_2 \; p_3$

$a_3 : p_1 \; p_3$

$M = \{a_1 p_2, \; a_2 p_3, \; a_3 p_1\}$ Claim: M is not popular.

$N = \{a_1 p_1, \; a_2 p_2, \; a_3 p_3\}$ $\quad M < N$

This lecture : preference lists are strict

Next : arb. pref lists (may contain ties)

G : instance (one-sided pref lists)

Q : Does G admit a pop. matching?

If yes, compute one. O/w can you declare?

may or may not

ToDo: construct an instance that does not admit a pop. matching

Q : G, M : Is M popular in G? [verification]

$a_1 : p_1 \quad p_2$

$a_2 : p_2 \quad p_3$

$a_3 : p_1 \quad p_3$

$a_3 \to p_1$ ✓

$a_1 \to p_1$ ✗

$a_1 \to p_2$ ✓

$a_2 \to p_2$ ✗

$a_2 \to p_3$ ✓

$M = \{ a_1 p_2, a_2 p_3, a_3 p_1 \}$

$a_1 \to p_1$

$a_2 \to p_2$

$a_3 \to p_1$

Characterization of PMs [one-sided]

G : instance. M is a PM in G iff

a_i : "last-resort" post l_i & append it to the pref list of a_i

$G \longrightarrow G$ augmented with last-resort posts

$\forall a \in A$: $f(a)$ = rank-1 post for a = unique & well-defined [strict]

p : f-post if $\exists a$ s.t. $p = f(a)$

for f-post p : $f(p) = \{ a \mid p = f(a) \}$

$s(a)$ = top-pref post for a that is not an f-post

		$f(\cdot)$	$s(\cdot)$
a_1 :	$p_1 \; p_2 \; \ell_1$	p_1	p_2
a_2 :	$p_1 \; \ell_2$	p_1	ℓ_2

f-posts = $\{p_1\}$ $\quad f(p_1) = \{a_1, a_2\}$

		$f(\cdot)$	$s(\cdot)$
a_1 :	$p_1 \; p_2 \; \ell_1$	p_1	ℓ_1
a_2 :	$p_2 \; p_3 \; \ell_2$	p_2	p_3
a_3 :	$p_1 \; p_3 \; \ell_3$	p_1	p_3

f-posts = $\{p_1, p_2\}$

$f(p_1) = \{a_1, a_3\}$, $\quad f(p_2) = \{a_2\}$

$s(a)$ is well-defined for every a

Recall: strict pref. list [Abraham et. al. '2007]

Claim 1 $\quad$ G: one-sided pref list instance.

M is a p.M. in G iff

1. Every f-post p is matched in M s.t. $M(p) \in f(p)$
2. $\forall a \in A$, $M(a) \in \{f(a), s(a)\}$

If $M(a_i) = \ell_i \Rightarrow a_i$ is unmatched in M (wrt G)

Pf. of Claim 1 :

To show : If M is a P.M. then (1) holds.

Pf : Sps. not. $\Rightarrow \exists$ f-post p that is either unmatched or $M(p) \notin f(p)$

Goal: Construct N s.t. $N \succ M$

$N \succ M$: suffices

N need not be a pop. matching in G

Sps. p is unmatched. Pick $a \in f(p)$

$$N = M \setminus \{(a, M(a))\} \cup \{(a, p)\}$$

then $N \succ M \Rightarrow \perp \because M$ is a pop. matching

p is matched but $M(p) \notin f(p)$

$b = M(p)$, $a \in f(p)$

(i) $f(b)$ is free in M

$$N = M \setminus \{\{b, M(b)\}\} \cup \{(b, f(b))\} \setminus \{(a, M(a))\}$$
$$\cup \{(a, p)\}$$

$N \succ M$ because a & b both prefer N over M & others are indifferent $\Rightarrow \perp$

(ii) $f(b)$ is matched $\Rightarrow$ $c = M(f(b))$

$$N = M \setminus \{(c, M(c))\}$$

$$\{\{b, M(b)\}\} \cup \{(b, f(b))\} \setminus \{(a, M(a))\}$$
$$\cup \{(a, p)\}$$

$N \succ M \Rightarrow \perp$